CHEBYSHEV COVERS AND EXCEPTIONAL NUMBER FIELDS
(PRELIMINARY VERSION)

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1. Introduction

We have recently found a collection of rational functions which are very remarkable from the point of view of Grothendieck’s dessins d’enfants. Some of the fibers of these rational functions yield number fields which are likewise very remarkable from the point of view of algebraic number theory. This preliminary version of the paper describes our results, sometimes precisely, sometimes sketchily. The final version of this paper will give precise statements and proofs within the organizational structure set up here.

The Chebyshev covers of our title are the rational functions

\[ T_{m,n}(x) = \frac{T_{m/2}(x)^n}{T_{n/2}(x)^m}, \]

\[ U_{m,n}(x) = \frac{U_{m/2}(x)^{2n}}{U_{n/2}(x)^{2m}}, \]

indexed by positive integers \( m \) and \( n \). Here \( T_w(x), U_w(x) \in \mathbb{Z}[x, \sqrt{x+2}, \sqrt{x-2}] \) are slightly modified versions of the classical Chebyshev polynomials as explained in Section 2. Square roots cancel so that \( T_{m,n}(x) \) and \( U_{m,n}(x) \) are always in \( \mathbb{Q}(x) \). The theory quickly reduces to the cases where \( m \) and \( n \) are relatively prime with \( m < n \) and, in the \( U \) case, not both odd. Henceforth we restrict to these cases. We use the word “cover” because our main point of view is that the \( T_{m,n} \) and \( U_{m,n} \) are functions from a Riemann sphere with coordinate \( x \) to another Riemann sphere with coordinate \( s \).

Sections 3-8 concern facts about Chebyshev covers. Their main critical values are \( s = 0, s = 1, \) and \( s = \infty \). Besides the obvious critical points in the preimages of these critical values, there are \( \lfloor (k-1)/2 \rfloor \) other critical points with \( k \) always \( n - m \) throughout this paper. Our first main result about Chebyshev covers is that the bulk of their bad reduction—all of it if \( k = 1,2 \)—is at primes dividing \( mn \). Our second main result, stated in the greater generality of quasiChebyshev covers, is that for \( m > 1 \), the monodromy group is the entire alternating or symmetric group on its degree. This combination of properties has not been previously exhibited in such high degrees. The closest comparison would be the ABC covers \( A_{m,n}(x) = k^{-k} x^m (n - mx)^k \), relevant here for exactly the same range of \( (m, n) \). The \( A_{m,n} \) have singular values within \( \{0, 1, \infty\} \) and have been studied in many places. For example, [1] focuses on the singular fiber \( A_{m,n}^{-1}(1) \). The \( A_{m,n} \) have bad reduction only at primes dividing \( mnk \), and monodromy group the full symmetric group. However the degree of \( A_{m,n} \) is only \( n \), while the degree of our \( T_{m,n} \) is \( mn/2 \) or \( mn(n-1)/2 \) and the degree of our \( U_{m,n} \) is \( m(n-1) \).
Let
\[ T_{m,n}(x) = \frac{T_{m,n}(0,x)}{T_{m,n}(\infty,x)}, \quad U_{m,n}(x) = \frac{U_{m,n}(0,x)}{U_{m,n}(\infty,x)}, \]
in lowest terms, with monic numerator and denominator in \( \mathbb{Z}[x] \). For \( s \in \mathbb{C} \), let
\begin{align*}
T_{m,n}(s,x) &= T_{m,n}(0,x) - sT_{m,n}(\infty,x), \\
U_{m,n}(s,x) &= U_{m,n}(0,x) - sU_{m,n}(\infty,x).
\end{align*}
Then the fibers above \( s \) are given as the roots of the corresponding polynomials. The algebras \( \mathbb{Q}[x]/T_{m,n}(s,x) \) and \( \mathbb{Q}[x]/U_{m,n}(s,x) \) are number fields for generic \( s \in \mathbb{Q} \).

Sections 9-11 concern these number fields and closely related ones. Section 9 says that field discriminants of \( T_{m,n}(s,x) \) and \( U_{m,n}(s,x) \) are very regularly behaved. Section 10 says that Galois groups of \( T_{m,n}((-1)^k,x) \) and \( U_{m,n}(1,x) \) are smaller than expected by genericity, but otherwise Galois groups of specializations generally agree with their obvious upper bound. Section 11 gives specific examples of number fields drawn from settings where \( m \) and \( n \) are close prime powers. From \( T_{8,9} \) and \( U_{8,9} \), we get polynomials with discriminant of the form \( \pm 2^a 3^b \) and Galois group the full alternating or symmetric group on the degree. The many such fields in [3] have degree up through 33 while ours here include degrees 35, 36, and 64. The degree 64 fields have sufficiently high degree for the set of ramifying primes \( \{2, 3\} \) that they are exceptional in the technical sense of [4]. The polynomial \( T_{25,27}(1,x) \) has degree 300 and discriminant \( 3^{8945600} \). We find a degree 100 subfield of \( \mathbb{Q}[x]/T_{25,27}(1,x) \) with Galois group \( A_{100} \) and field discriminant of the form \( 3^a 5^b \), thus another exceptional number field. From \( T_{125,128} \) and \( U_{125,128} \), we expect five more exceptional fields with field discriminants of the form \( \pm 2^a 5^b \) and degrees up to 15875. However here the degrees are too high for the expected Galois group to be computationally confirmed.

Readers interested in getting as quickly as possible to exceptional number fields can simply check our conventions with regard to Chebyshev polynomials in Section 2 and then skip immediately to Section 11. The sections in between can be reasonably read in many orders. In particular, Sections 6 and 7 form somewhat of a detour. These two sections are focused not on the objects of our title, but rather on their near environs in the theory of dessin d’enfants.

2. Chebyshev polynomials

We work in the biquadratic extension of \( \mathbb{Z}[x] \) obtained by adjoining \( \sqrt{x-2} \) and \( \sqrt{x+2} \). Although our main interest is in the interval \([-2, 2]\), we resolve square-root ambiguities by requiring that, as usual, \( \sqrt{x-2} \) and \( \sqrt{x+2} \) are positive on \((2, \infty)\). We define these quantities on all of \( \mathbb{R} \) by analytically continuing on the upper half plane only.

Let \( w \in \{1/2, 1, 3/2, 2, \ldots \} \). Our Chebyshev polynomials of the first and second kind respectively are \( T_w(x) \) and \( U_w(x) \) where
\[ T_w(z + 1/z) = z^w + z^{-w}, \quad U_w(z + 1/z) = z^w - z^{-w}. \]
These Chebyshev polynomials factor canonically into their interior and boundary parts
\begin{align*}
T_w(x) &= t^*_w(x)t_w(x), \\
U_w(x) &= u^*_w(x)u_w(x).
\end{align*}
Here the boundary parts \( t_w^*(x) \) and \( u_w^*(x) \) depend only on their index \( w \) modulo one and are
\[
t_0^*(x) = 1, \quad t_{1/2}^*(x) = \sqrt{x + 2}, \quad u_1^*(x) = \sqrt{x^2 - 4}, \quad u_{1/2}^*(x) = \sqrt{x - 2}.
\]
The interior parts are monic polynomials in \( \mathbb{Z}[x] \) and have all roots in \((-2, 2)\). Explicitly, if \( w \) is integral,
\[
t_w(x) = \sum_{j=0}^{\lfloor w/2 \rfloor} \frac{(-1)^j}{w-j} \binom{w-j}{j} x^{w-2j}, \quad u_w(x) = \sum_{j=0}^{\lfloor (w-1)/2 \rfloor} (-1)^j \binom{w-1-j}{j} x^{w-1-2j}.
\]
If \( w \) is half-integral,
\[
t_w(x) = u_{w+1/2}(x) - u_{w-1/2}(x), \quad u_w(x) = u_{w+1/2}(x) + u_{w-1/2}(x).
\]
One should view the \( T_w(x) \) and \( U_w(x) \) as indexed by degree. Here boundary roots in \( \{-2, 2\} \) count with multiplicity one half while interior roots in \((-2, 2)\) count with multiplicity one, in accordance with the presence of square roots. One has a variety of formulas connecting the Chebyshev polynomials, many direct translations of formulas more widely known in the context of cyclotomy and/or trigonometry. The connection between our Chebyshev polynomials and the most traditional ones is
\[
t_w(x) = 2T_w^{tr}(x/2), \quad u_w(x) = U_w^{tr}(x/2)
\]
in the integral case. Note in particular the index shift in the case of Chebyshev polynomials of the second kind. Our notation places the focus on \( U_w(x) = u_w^*(x) u_w(x) \) which does indeed have degree \( w \). It also emphasizes primes of bad reduction as \( \text{disc}(u_w(x)) = 2^{w-1}w^{w-3} \).

### 3. Chebyshev covers

The Chebyshev polynomials having been defined, the definition of Chebyshev covers given in (1.1) is now complete. Basic facts about them can be established by direct computation. When possible, we treat the two cases simultaneously, writing \( F \) for either \( T \) or \( U \) throughout this paper.

First, we explain how the excluded cases reduce to the considered cases. Suppose briefly that \( m = m'd \) and \( n = n'd \) for \( d > 1 \). Then
\[
F_{m,n}(x) = F_{m',n'}(T_d(x))^d.
\]
Similarly,
\[
U_{m,n}(t^2, x) = -T_{m,n}(t, -x)T_{m,n}(-t, -x).
\]
whenever \( m \) and \( n \) are both odd. Finally, and immediately from (1.1),
\[
F_{m,n}(x) = F_{n,m}(x)^{-1}.
\]
The standing assumption \( m < n \) breaking the \( m \leftrightarrow n \) symmetry is particularly convenient since phenomena becomes associated to only \( m \) and others to only \( n \). For example, in Section 6, vertices of valence related to \( n \) behave simply while those related to \( m \) behave in a complicated way. In Section 9, in reverse, primes dividing \( m \) behave more simply than primes dividing \( n \).

The details of all our considerations depend on the parity of \( m \) and \( n \). Often we must therefore break into five cases, naturally denoted \( T01, T10, T11, U01, \)
and $U10$, the case $U11$ having been excluded. As an example of a case distinction, numerator and denominator in (1.1) are relatively prime in cases $T01$ and $T10$, but there is a cancellation in the remaining three cases.

The zeros of $T_{m,n}$ and $U_{m,n}$ from left to right have multiplicity

$$
(m, n) = (0, 1) : \begin{align*}
\lambda_{m,n}^T &= n^{m/2}, \\
\lambda_{m,n}^U &= n, (2n)^{m/2-1}, k,
\end{align*}
(3.4)$$

$$
(1, 0) : \begin{align*}
\lambda_{m,n}^T &= \frac{n^2}{2}, n^{(m-1)/2}, \\
\lambda_{m,n}^U &= (2n)^{(m-1)/2}, k.
\end{align*}
(1, 1) : \begin{align*}
\lambda_{m,n}^T &= \frac{k^2}{2}, n^{(m-1)/2},
\end{align*}
$$

The poles of $T_{m,n}$ and $U_{m,n}$ from left to right have multiplicity

$$
(m, n) = (0, 1) : \begin{align*}
\lambda_{m,n}^T &= \frac{m}{2}, m^{(n-1)/2}, \\
\lambda_{m,n}^U &= (2m)^{(n-1)/2},
\end{align*}
(3.5)$$

$$
(1, 0) : \begin{align*}
\lambda_{m,n}^T &= m^{n/2}, \\
\lambda_{m,n}^U &= m, (2m)^{n/2-1}.
\end{align*}
(1, 1) : \begin{align*}
\lambda_{m,n}^T &= m^{(n-1)/2},
\end{align*}
$$

These zeros and poles are all in $[-2, 2]$ and so divide into interior singularities in $(-2, 2)$ and boundary singularities in $\{-2, 2\}$. Always interior zeros have multiplicity $n$ in Case $T$ and $2n$ in Case $U$. Always interior poles have multiplicity $m$ in Case $T$ and $2m$ in Case $U$. The case distinctions are important only for boundary singularities, of which there are always one in Case $T$ and two in Case $U$. Our appropriation of the classical terminology of Chebyshev polynomials has an extra virtue not present in the classical theory: general speaking our functions $T_{m,n}$ and $U_{m,n}$ are similar; the main difference is that our functions of the second kind involve an extra factor of two in many ways.

While the left-to-right order in (3.4) and (3.5) is certainly of fundamental importance, often only $\lambda_{m,n}^{F,\sigma}$ as a partition of the degree enters into a given consideration. As partitions, $\lambda_{m,n}^{F,0}$ and $\lambda_{m,n}^{F,\infty}$ belong to a triple, the third member being the partition $\lambda_{m,n}^{F,1}$ giving the multiplicities of the preimages of 1. This third partition is $m1\cdots1$ except when $k$ is a multiple of six, in which case it is $m21\cdots1$ as discussed below. Here $m$ is the multiplicity of $\infty$ in the preimage of 1. It reflects that $F_{m,n}(1, x)$ always has degree $m$ less than $F_{m,n}(s, x)$, a fact which can be easily checked. In comparison to the other parts of our three partitions, the many 1’s in $\lambda_{m,n}^{F,1}$ enter differently into our considerations. First, the corresponding roots are not critical points, exactly because their multiplicity is 1 rather than some larger number. Second, most of these roots are not real.

In this preliminary version of the paper we generally avoid going into cases. Rather we systematically illustrate general results with the particular cover

$$
T_{8,9}(x) = \frac{(x^4 - 4x^2 + 2)^9}{(x - 1)^8(x + 2)^4(x^3 - 3x - 1)^8}.
$$

Also we concentrate on the case $T01$ this cover represents. Figure 3.1 plots $T_{8,9}(x)$. The zeros on this plot interlace with the poles. This interlacing always occurs when $k = 1, 2$. It is usually not the case in general, as there are approximately $k/2$ more poles than zero. Rather the geometric situation is substantially more complicated because of the presence of approximately $k/2$ critical points, which we discuss next.

In Case $T01$, the $m/2$ zeros of multiplicity $n$ each yield a contribution of $(n-1)$ to the critical divisor, for a total contribution of $(mn - m)/2$. The $(n-1)/2$ poles of
multiplicity $m$ and the further pole at $x = -2$ of multiplicity $m/2$ similarly yield a total contribution of $(mn - n - 1)/2$ to the critical divisor. The point $\infty$ contributes $m - 1$. The critical divisor of a rational function of degree $N$ always has degree $2N - 2$, which here is $mn - 2$. This shows that there are $(n - m - 1)/2 = (k - 1)/2$ remaining critical points to be found. Similar simple computations for the other cases reveal that in general there are $\lfloor (k - 1)/2 \rfloor$ remaining critical points to be found.

A first take on this situation is that the cases $k = 1, 2$ are worth pursuing while the cases $k > 2$ are not. Indeed we will focus on the cases $k = 1, 2$. However, arbitrary $k$ is in fact a natural context. One argument for this is simply that our highest degree examples in Section 11 come from the setting $k = 3$. However a much more structural reason is given in Section 9: in the study of the bad reduction of a given $F_{m,n}$, other covers $F_{m',n}$ and $F_{m,n'}$ enter with very different index differences.

By taking the derivative of $F_{m,n}$ we find that these remaining critical points depend only on $k$, being always the roots of $u_{k/2}(x)$. We find that the corresponding critical values depend on whether one is in Case $T$ or Case $U$, but again depend on $m$ and $n$ only through the difference $k = n - m$. Taking $m = 1$ and $n = k + 1$ to get the simplest formula, these critical values are the roots of the polynomials

$$d_T^k(s) = \pm \text{Res}_x \left( (x + 2)^{[k/2]} - st_{(k+1)/2}(x), u_{k/2}(x) \right),$$
$$d_U^k(s) = \pm \text{Res}_x \left( (x - 2)^k - s(x + 2)^{\delta} u_{(k+1)/2}(x)^2, u_{k/2}(x) \right).$$

Here $\delta$ either zero or one according to whether $k$ is even or odd, and the sign is chosen so that $d_T^k(s)$ is always monic. The first few of these polynomials in factored
form are

\[ d_1^T(s) = 1, \quad d_1^U(s) = 1, \]
\[ d_2^T(s) = 1, \quad d_2^U(s) = 1, \]
\[ d_3^T(s) = s + 1, \quad d_3^U(s) = s + 27, \]
\[ d_4^T(s) = s + 4, \quad d_4^U(s) = s - 16, \]
\[ d_5^T(s) = s^2 + 11s - 1, \quad d_5^U(s) = s^2 + 625s + 3125, \]
\[ d_6^T(s) = (s - 1)(s - 27), \quad d_6^U(s) = (s - 1)(s - 729). \]

One has several recurring patterns among these polynomials, including in both cases that 1 is a root if and only if \( k \) is a multiple of 6. Thus usually one has \( \lfloor (k - 1)/2 \rfloor \) critical values beyond \( \{0, 1, \infty\} \) but if \( k \) is a multiple of six, one has \( \lfloor (k - 1)/2 \rfloor - 1 = k/2 - 2 \) extra critical values and the ramification partition for 1 takes the shape \( m21^{N-m-2} \) rather than \( m1^{N-m} \) as mentioned above.

From the obvious critical points we knew the poles of \( F'_{m,n}(x) \) and some of the zeros. We have just found the remaining zeros. To determine the derivative completely, we need only the constant in the factorization constant \( \cdot \) monic/monic. This constant is \( -mn \) in Case \( T \) and \( -2mn \) in Case \( U \). Our derivative calculation is valid in all characteristics and reveals that geometric situation looks much the same when reduced modulo primes not dividing \( mn \), but very different when reduced modulo primes dividing \( mn \).

4. Discriminant Formulas

Our formulas for the discriminant of Chebyshev covers play a central role in our study so we state them here in all cases. The degree \( N_{F_{m,n}} \) of \( F_{m,n} \) enters repeatedly into our formulas. We write the degree as

- Cases \( T01, T10 : \quad A = mn/2, \)
- Cases \( U01, U10 : \quad C = m(n - 1). \)
- Case \( T11 : \quad B = m(n - 1)/2, \)

to simplify the notation.
For $s \neq 1$ we have,
\[\text{T01: } \text{disc}(T_{m,n}(s,x)) = (-1)^{\Delta(m/2)+\Delta((n+1)/2)}2^{A-m/2-n}m^A n^A (s-1)^{m-1} s^{-m/2} d_k^T(s),\]
\[\text{T10: } \text{disc}(T_{m,n}(s,x)) = (-1)^{(m-1)n/4}2^{A-m-n/2}m^A n^A (s-1)^{m-1} s^{A-m/2-1/2} d_k^T(s),\]
\[\text{T11: } \text{disc}(T_{m,n}(s,x)) = (-1)^{(m-1)(n-1)/4}m^B n^{B+(k-2)m-k} (s-1)^{m-1} s^{B-(m+1)/2} d_k^T(s),\]
\[\text{U01: } \text{disc}(U_{m,n}(s,x)) = 2C m^{C_n} n^{C+(2k-2)m-k} (s-1)^{m-1} s^{C-m/2} d_k^U(s),\]
\[\text{U10: } \text{disc}(U_{m,n}(s,x)) = (-1)^{(m+n+1)/2}2^{C-k} m^{C_n} n^{C+(2k-2)m-k} (s-1)^{m-1} s^{C-m/2-1/2} d_k^U(s).\]

The main new content here is the exponents on the arithmetic bases $-1, 2, m,$ and $n$. The exponents on the main geometric factors $s$ and $s-1$ and also the presence of the secondary geometric factor $d_k^F(s)$ were known from the previous section. We prove our discriminant formulas by methods similar to those used in [3], using again Equations (7.13)-(7.14) there as a starting point.

For $s = 1$, we indicate degree by $a = A - m$, $b = B - m$, and $c = C - m$. In the same order of cases as before, we have the following complementary statements.

\[\text{disc}(T_{m,n}(1,x)) = (-1)^{\Delta(m/2-1)+\Delta(n/2-3/2)}2^{a-n/2-k/2}m^a n^{a-1} d_k^T(1),\]
\[\text{disc}(T_{m,n}(1,x)) = (-1)^{(m-1)(n+2)/4}2^{a-n/2}m^a n^{a-1} d_k^T(1),\]
\[\text{disc}(T_{m,n}(1,x)) = (-1)^{(m-1)(n+1)/2}m^b n^{b+(k-2)m/2-k/2} d_k^T(1),\]
\[\text{disc}(U_{m,n}(1,x)) = (-1)^{m/2}2^{c-1}m^c n^{c-1-k} d_k^U(1),\]
\[\text{disc}(U_{m,n}(1,x)) = (-1)^{n/2}2^{c-1}m^c n^{c-1-k} d_k^U(1).\]

Similarly we have formulas for the discriminant of the separable part of $F_{m,n}(\sigma,x)$ when $\sigma$ is a root of $d_{m,n}^F(s)$.

5. Dessins and monodromy

We sometimes use $-\infty$ as a synonym for the point $\infty$ in the base projective line $\mathbb{P}_s^1(C)$ when it seems more communicative. The geometric dessin $D_{m,n}^F$ of a cover $F_{m,n}$ is $F_{m,n}^{-1}([-\infty,0])$ considered as a subset of $\mathbb{P}_s^1(C)$. Until the last paragraph we assume $k \leq 2$.

Figure 5.1 draws the geometric dessin $D_{9,9}^T$. The rightmost point of this dessin on the real line is the pole $2 \cos(\pi/9) \approx 1.88$. The next rightmost point on the real line is the zero $2 \cos(\pi/8) \approx 1.85$. These two parts are connected with eight edges.
Figure 5.1. The dessin $T_{8,9}^{-1}([\infty, 0])$ drawn in the region $[-2.1, 2.1] \times [-0.45, 0.45]$ of the complex $x$-plane. The five poles of are interspersed with four zeros connected by edges in accordance with the diagram (5.1). The roots of $T_{8,9}(-1, x)$ mark the centers of the thirty-six edges while the roots of $T_{8,9}(1, x)$ mark the centers of the twenty-eight bounded faces.

Let $\Gamma_{8,9}^T$ be $D_{8,9}^T$ considered as combinatorial dessin. So $D_{8,9}^T$ is a specific subset of $C$, while $\Gamma_{8,9}^T$ a planar graph which is allowed to be slid around freely. In practice, geometric dessins are computer drawings, while combinatoric dessins are more truly a children's drawings. The distinction is fundamental, as indeed one often views combinatorial dessins as the input to the theory and geometric dessins as the output. Nonetheless, we normally say simply dessin, as the context is clear.

Besides geometric dessins $D$ and combinatorial dessins $\Gamma$, a closely related third object $\gamma$ comes into play. We call this third object the reduced combinatorial dessin, or again just dessin. It is constructed from $\Gamma$ by iteratively identifying two edges which together bound a face, losing also the bounded face in the process. The lost face corresponds to a non-critical point in the fiber $F^{-1}(1)$. Collapsing edges two at a time in this way, many edges can be collapsed to one, and $\gamma$ is to be viewed as a bipartite weighted planar graph. The weight of a vertex of $\gamma$ is the valence of that vertex in $\Gamma$. The weight on an edge of $\gamma$ is the number of edges in $\Gamma$ reducing to it. So edge weights determine vertex weights. However one often keeps the focus on vertex weights since they are the more basic quantities.

In our example, the reduced combinatoric dessin $\gamma_{8,9}^T$ is

\[
\gamma_{8,9}^T = 4 - 9 - 8 - 9 - 8 - 9 - 8 - 9 - 8 - 9 - 8.
\]

Here and in the sequel, numbers in bold are multiplicities of poles, while numbers in regular type are multiplicities of zeros.

Since the partition $\lambda^{F,1}_{m,n}$ controlling faces is $m1 \cdots 1$, it is clear from the definition that the graph $\gamma_{m,n}^F$ is a tree in general. We prove that, as one might expect from the example (5.1), that $\gamma_{m,n}^F$ is in fact always a segment. The proof is elementary, using the fact that all vertices are real.

As always for dessins, the monodromy group is generated by operators $g_0$ and $g_\infty$ acting on the edges of $\Gamma$ by rotating a given edge minimally counterclockwise about the endpoint which is a zero or pole respectively. The highly structured nature of
our dessins lets us label edges of \( \Gamma \) in simple ways so that the monodromy action can be written down in algebraic terms.

The dessin \( \mathcal{D}_{F, m,n} \) brings to visual prominence two polynomials of particular interest. First, it is reasonable to call the \( N \) roots of \( F_{m,n}(-1, x) \) the centers of the \( N \) edges. Similarly, it is reasonable to view the \( N - m \) roots of \( F_{m,n}(1, x) \) as the centers of the bounded faces. In particular, we have a well-defined way to label roots, even though the analytic fact that they lie approximately in columns is not confirmed. We use this labeling in Section 10.

One can think of dessins more dynamically, viewing \( s \) as representing time and the dessin as traced out by \( N \) moving points, distinguishable in the interval \((-\infty, 0)\). In our cases, the particles start out at time \( s = -\infty \) clumped at the approximately \( n/2 \) poles. They expand in circles of generic size \( m \) or \( 2m \) about the fixed poles until approximately \( s = -1 \) where circles about boundary poles have moved inward to approximately vertical lines while circles about interior poles have split into two approximately vertical columns. Then interior columns pair with their other adjacent column, and the process reverses as the points contract in circles of generic size \( n \) or \( 2n \) to the approximately \( m/2 \) zeros at \( s = 0 \).

For general \( k \), the dynamic description just given goes through in large part. The difference is that a pair of real roots can coalesce to become a pair of conjugate non-real roots and one has to make choices to label roots consistently over all of \((-\infty, 0)\). The extra coalescence is necessary for a consistent picture to account for the approximately \( n/2 \) initial circles becoming approximately only \( m/2 \) circles.

6. QuasiChebyshev covers

In this section, we restrict to the cases \( k = 1, 2 \) and thus three point covers. We define a three point cover to be a quasiChebyshev cover if its ramifications partitions over 0, 1, and \( \infty \) agree with a Chebyshev cover and if it’s normalized in the same way as a Chebyshev cover, as we’ll explain. The set \( \mathcal{F}_{m,n} \) of quasiChebyshev covers agreeing numerically with a given cover \( F_{m,n} \) is finite and can be computed by solving equations.

In Case T01, we begin our normalization by requiring that a quasiChebyshev cover send \( \infty \) to 1 and \(-2\) to \( \infty \) just as \( T_{m,n} = T_{m,m} + 1 \) itself does. Then we have the general form

\[
T_{m,n}^{\text{gen}}(x) = \frac{\left(x^{m/2} + \sum_{i=1}^{m/2} a_i x^{m/2-i}\right)^n}{(x + 2)^{m/2} \left(x^{m/2} + \sum_{i=1}^{m/2} c_i x^{m/2-i}\right)^m}.
\]

We look at the “new factor” \( \Delta_{T_{m,n}}(x) \) of the numerator of the derivative of \( T_{m,n}^{\text{gen}}(x) \). The rational functions we seek are those for which \( \Delta_{T_{m,n}}(x) \) is reduced to a constant.

An affine transformation fixes \(-2\) if and only if it has the form \( x \mapsto \lambda x + (2\lambda - 2) \).

If we have a solution then changing \( x \) by this affine transformation gives another solution. There are two cases to distinguish. The highest term of \( \Delta_{T_{m,n}}(x) \) is \((m+1)a_1 - m(c_1 + 1))x^m\). Either \( a_1 \) and \( c_1 \) are both \( m \), or they are both different from \( m \). In the former case, the three point covers we construct have non-trivial automorphisms while in the latter case they do not. We focus on the latter case first, which is the main case. In this main case, we complete our normalization by requiring \( a_1 = 0 \) or equivalently \( c_1 = -1 \).
In our continuing example, we have

\[
\Delta_{8,9}^T(x) = (-24 - 18a_2 + 16c_2)x^6 \\
(12a_2 - 27a_3 - 36a_4 - 2a_2c_2 + 56c_3 + 32c_4)x^4 \\
(30a_3 - 36a_4 + 4a_2c_2 - 11a_3c_2 + 6a_2c_3 + 72c_4)x^3 \\
(48a_4 - 14a_3c_2 - 20a_4c_2 + 20a_2c_3 - 3a_3c_3 + 14a_2c_4)x^2 \\
(-32a_4c_2 + 2a_3c_3 - 12a_4c_3 + 36a_2c_4 + 5a_3c_4)x \\
(-16a_4c_3 + 18a_3c_4 - 4a_4c_4).
\]

Equating the coefficients of \(x^6, x^5, x^4,\) and \(x^3\) successively to zero gives

\[
c_2 = (12 + 9a_2)/8, \\
c_3 = (-20 - 9a_2 + 9a_3)/8, \\
c_4 = (560 + 216a_2 + 9a_2^2 - 144a_3 + 144a_4)/128, \\
a_4 = (-112 - 40a_2 - a_2^2 + 24a_3 + 2a_2a_3)/16.
\]

Writing \(a = a_2\) and \(b = a_3\) we then have

\[
\frac{256}{9} \Delta_{8,9}^T(x) = 8 \left(5a^3 - 3ba^2 + 60a^2 - 48ba + 48a - 12b^2 + 48b - 448\right) x^2 + \\
4 \left(10a^3 + 22ba^2 + 120a^2 - 7b^2 a + 72ba + 544a - 108b^2 + 320b + 896\right) x + \\
(ba^3 - 40a^3 - 2b^2 a^2 + 156ba^2 - 1696a^2 - 24b^2 a + 2128ba - 8320a - 576b^2 + 4544b - 10752)
\]

The variable \(b\) occurs quadratically, so we can not eliminate it by such elementary algebra. Instead, we take the resultant of the coefficients \(h_2(a,b)\) and \(h_1(a,b)\) of \(x^2\) and \(x\) respectively in (6.1) to get a constant times

\[
g_{8,9}^T(a) = (a + 4) \left(35a^7 + 2380a^6 + 38192a^5 + 236480a^4 + 928000a^3 + 3015680a^2 - 3993600a - 16546214\right).
\]

The septic polynomial on the right has Galois group \(S_7\) and field discriminant \(-24 \cdot 3^5 \cdot 5^6 \cdot 7^2 \cdot 11^5 \cdot 19^3\). Each root \(\alpha\) of \(g_{8,9}(a)\) determines a quasiChebyshev cover \(T_{m,n}^\alpha(x)\) with \(\alpha = -4\) yielding the Chebyshev cover \(T_{8,9}(x)\).

In general, the part of \(T_{m,m+1}\) consisting of covers without extra automorphisms is likewise indexed by the roots of a suitable moduli polynomial \(g_{m,m+1}^T(a)\). The rest of \(T_{m,m+1}\), as we’ll see topologically, is indexed by divisors \(d\) of \(m\). Besides \(1\) and roots \(a\) of another moduli polynomial \(g_{m,n}^T[d]\) so that \(T_{m,m+1}^{[d,a]}\) has exactly \(d\) automorphisms. The cases \(T_{10}, T_{11}, U_{01}\) and \(U_{11}\) are easier in that there are no quasiChebyshev polynomials with extra automorphisms.

We have computed all cases up through \(|\mathcal{F}_{m,n}| = 42\) as listed on Table ??.

In this range, the polynomial \(g_{m,n}^T[a]\) always factors over \(\mathbb{Q}\) into a linear factor corresponding to the Chebyshev cover \(F_{m,n}\) and a complementary irreducible factor corresponding to the other quasiChebyshev covers without extra automorphisms, just like in (6.2). Always the Galois group of the complementary factor is the full symmetric group on its degree. Always, except for very low degrees, the moduli polynomial is ramified at primes beyond those dividing \(m\) and \(n\). For
example, the field discriminant of the degree thirty four polynomial for $U_{8,9}^{\text{gen}}$ is $27^{1}3^{4}45^{2}7^{2}11^{2}13^{1}19^{3}23^{1}29^{1}31^{3}37^{4}47^{3}$. The three point covers themselves can be further ramified beyond the ramification in the moduli polynomial. For example, $T_{8,9}^{[2]}$, with equation given in Figure 7.1, is defined over $\mathbb{Q}$ but has bad reduction at 5 and 7 as well as at 2 and 3. In short, in the collection of quasiChebyshev covers only the Chebyshev covers seem to be arithmetically special.

One could go further in relating our covers to other covers as follows. One can demand that $\Delta_{m,n}^{T}$ be simply linear, rather than constant, thereby generically seeking covers with ramification partitions $(\lambda_0^{m,n}, (m-1)1 \cdots 1, \lambda_{\infty}^{m,n})$ above $(0, 1, \infty)$ and a fourth unspecified ordinary ramification point. The solution set $F_{m,n}^{1}$ is a curve containing $F_{m,n}$. Thus this approach embeds the finite set $F_{m,n}$ in a single connected family. In our example case, $T_{1}^{m,n}$ is the elliptic curve of conductor $1210 = 2 \cdot 5 \cdot 11^2$ defined by $h_2(a, b) = 0$.

We have not yet used the constant term $h_0(a, b)$ of $\Delta_{8,9}^{T}(x)$. However, using the same notation to indicate the general case, the root $x_{\text{crit}} = -h_0(a, b)/h_1(a, b)$ of the linear polynomial $\Delta_{m,n}^{F}(x)$ can be viewed as a function on the curve $F_{m,n}^{1}$. Its critical value $s_{\text{crit}} = F_{m,n}^{\text{gen}}(x_{\text{crit}})$ can likewise be viewed as a function on $F_{m,n}^{1}$. In fact, the function $s_{\text{crit}}$ presents $F_{m,n}$ as a three point cover of the line with coordinate $s_{\text{crit}}$. Our set $F_{m,n}$ is in the fiber above $\infty$. The rest of the fiber above $\infty$ and the entire fibers above 0 and 1 include other sets analogous to $F_{m,n}$, indexing three point covers of different partition triples. Already our example case $T_{8,9}$ is complicated, but lower degree cases which satisfy $F_{m,n}^{1} \cong \mathbb{P}^1$ are computationally easy. Even in this enlarged context, the Chebyshev covers seem to be the only three point covers which are arithmetically special.

7. QuasiChebyshev Dessins

In this section, we again restrict to the cases $k = 1, 2$. We explain how “half” of the topological simplicity of Chebyshev covers is kept by quasiChebyshev covers. This allows us to index the sets $F_{m,n}$ of quasiChebyshev covers in a particularly simple way.

A quasiChebyshev cover has a dessin, again simply the preimage $D$ of $[-\infty, 0]$. It has a combinatoric dessin $\Gamma$ and a reduced combinatoric dessin $\gamma$, respectively a planar bipartite graph and a planar bipartite weighted tree. At issue is to say explicitly what the possibilities for $\gamma$ are.

The vertex weights on $\gamma$ are given, being by definition exactly the same as in the Chebyshev case. The sum of the weights of edges incident on a vertex is exactly the given vertex weight. Vertex weights then completely determine edge weights, but many candidates for $\gamma$ yield zero or negative edge weights. For example, the weighted bipartite tree

$$
\begin{array}{ccccccc}
8 & -9 & 1 & 3 & 6 & -8 & 2 \\
8 & -9 & 4 & -9 & 8 & -8 & 2 \\
\end{array}
$$

is not a reduced combinatorial dessin because of the negative edge weight $-1$. Clearly for the partitions $\lambda_{\infty} = 48^4$ and $\lambda_0 = 9^4$ in this example, a weight nine
vertex can never have valence one. It then follows that it can never have valence $\geq 3$ either, and thus must have valence two.

In general, we say a zero-vertex is large if it has weight $n$ in Case $T$ or $2n$ in Case $U$. Similarly, a polar-vertex is large if has weight $m$ in Case $T$ or $2m$ in Case $U$. Otherwise vertices are medium or small, meaning half the generic weight or weight 1 respectively. There is one small or medium vertex in Case $T$ and two small or medium vertices in Case $U$, as mentioned in Section 3. The numerics force in a very simple way, illustrated for $T_{8,9}^{gen}$ above, that the large zero-vertices have valence two in all cases $T_{01}, T_{10}, T_{11}, U_{01},$ and $U_{10}$ and the medium or small zero-vertices have valence one. More surprisingly, in all cases, there is no condition on the polar vertices, whether they be large, medium, or small. For example, the weighted bipartite trees

$$
\begin{array}{c}
8 & 8 & 7 & 2 \\
5 & 9 & 4 & 8 \\
\end{array}
\quad
\begin{array}{c}
8 & 8 & 7 & 2 \\
1 & 9 & 8 & 8 \\
\end{array}
$$

are each reduced combinatorial dessins. They each have polar vertices of valence 1, 2, and 3.

Because edge weights are automatic from vertex weights, we don’t need to write them. Because all zero-vertices have valence one or two, we only need to draw in the former. We distinguish between small and medium zero-vertices by drawing nothing and $\ast$ respectively. This last distinction is important only in Case $U_{01}$, since only in this case is there both a small and medium zero-vertex. By this procedure, any reduced combinatorial dessin $\gamma$ gives rise to a combinatorial object $\delta$ which we call a polar combinatorial dessin. The reduction $\gamma \mapsto \delta$ is bijective, and we focus on the $\delta$’s.

\[ \delta = 8 - 8 - 4 - 8 - 8 \]

\[ T_{8,9}^{\delta}(x) = \frac{(z^2 - 24 z + 84)^9}{z^2 (z^2 - 27 z + 135)^5} \]

with $z = \frac{5}{3}(x + 2)^2$

\[ \delta = 8 - 4 - 8 \]

\[ T_{8,9}^{\delta}(x) = \frac{(z - 8)^9}{z(z - 9)^8} \]

with $z = 9(x + 2)^4$

**Figure 7.1.** The two polar dessins $\delta$ indexing quasiChebyshev dessins in $T_{8,9}$ with rotational symmetry, and for each a formula for $T_{8,9}^{\delta}$. 
We now systematically use the letter $e$ to index objects. We use polar dessins $\delta$ to distinguish quasiChebyshev covers from each other. Thus, in Case $T_01$, the elements of $\mathcal{T}_{2e,2e+1}$ are $\mathcal{T}_{2e,2e+1}^{\delta}$ where $\delta$ runs over planar trees with a marked vertex. In our continuing example $T_{8,9}^{\text{gen}}$, the possibilities for $\delta$ are given in Figures 7.2 and 7.1.

\begin{align*}
4 - 8 - 8 - 8 - 8 & \quad -4.000 \quad 4 - 8 - 8 \quad -2.124 \\
8 - 4 - 8 - 8 - 8 & \quad -10.954 \quad 8 - 4 - 8 \quad -8.045 \\
4 - 8 - 8 & \quad 1.997 \quad 4 - 8 - 8 \quad -47.962 \\
8 & \quad 8 \\
4 - 8 & \quad 0.456 - 5.119i \quad 4 - 8 \quad 0.456 + 5.119i \\
8 & \quad 8 - 8
\end{align*}

Figure 7.2. The polar dessins $\delta$ indexing quasiChebyshev dessins in $\mathcal{T}_{m,n}$ without rotational symmetry and for each the corresponding root of (6.2).

Our terminology “polar dessin” is self-explanatory in Case $T01$ as $\delta$ refers to poles only, not zeros. In the remaining cases, the term reflects the fact that $\delta$ refers to all the poles, and only the small and medium zeros, of which there is at most one of each type. Figure 7.3 illustrates our drawing conventions for polar dessins in each case.

Recall that the $e^{\text{th}}$ Catalan number is

$$(7.1) \quad C_e = \frac{1}{e + 1} \binom{2e}{e} = \frac{(2e)!}{(e + 1)!e!}.$$  

For $e = 1, \ldots, 5$, the corresponding Catalan numbers are 1, 2, 5, 14, 42 and asymptotically one has $C_e \sim 4^e / (\sqrt{\pi} e^{3/2})$. The usual statement is that Catalan numbers count rooted planar trees, meaning a planar tree together with a marked
\[ \gamma_{6,7}^T = 3 - 7 - 6 - 7 - 6 - 7 \quad \delta_{6,7}^T = 3 - 6 - 6 - 6 \]
\[ \gamma_{5,6}^T = 3 - 5 - 6 - 5 - 6 - 5 \quad \delta_{5,6}^T = \star - 5 - 5 - 5 \]
\[ \gamma_{5,7}^T = 1 - 5 - 7 - 5 - 7 - 5 \quad \delta_{5,7}^T = - 5 - 5 - 5 \]
\[ \gamma_{6,7}^U = 7 - 12 - 14 - 12 - 14 - 12 - 1 \quad \delta_{6,7}^U = \star - 12 - 12 - 12 \]
\[ \gamma_{5,6}^U = 5 - 12 - 10 - 12 - 10 - 1 \quad \delta_{5,6}^U = 5 - 10 - 10 - 1 \]

Figure 7.3. On the left, the reduced combinatoric dessin of Chebyshev covers representing each of the five cases. On the right, the corresponding polar dessins. The procedure for passing from a reduced combinatoric dessin \( \gamma \) to the corresponding polar dessin \( \delta \) does not use the linear structure present in these diagrams, and works in the quasi-Chebyshev context.

<table>
<thead>
<tr>
<th>Case</th>
<th>Description</th>
<th>Mass</th>
<th>Masses for ( e = 1, 2, 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T_{2e,2e+1} )</td>
<td>Vertex marked as medium</td>
<td>( C_e (e + 1) )</td>
<td>( 6, 1.5, 3.5, 8.75, 25.2, \ldots )</td>
</tr>
<tr>
<td>( T_{2e+1,2e+2} )</td>
<td>Medium half-edge</td>
<td>( C_e )</td>
<td>( 1, 2, 5, 14, 42, \ldots )</td>
</tr>
<tr>
<td>( T_{2e+1,2e+3} )</td>
<td>Small half-edge</td>
<td>( C_e )</td>
<td>( 1, 2, 5, 14, 42, \ldots )</td>
</tr>
<tr>
<td>( U_{2e+2,2e+3} )</td>
<td>Medium half-edge</td>
<td>( C_e (2e + 1) )</td>
<td>( 3, 10, 35, \ldots )</td>
</tr>
<tr>
<td>( U_{2e+1,2e+2} )</td>
<td>Small half-edge</td>
<td>( C_e (e + 1) )</td>
<td>( 2, 6, 20, \ldots )</td>
</tr>
</tbody>
</table>

Table 7.1. Description and masses of the sets \( F_{m,n} \). In each case the description is in terms of what needs to be added to a planar tree with \( e \) edges to get a polar dessin \( \delta \). The polar dessins \( \delta \) can have non-trivial rotational symmetry only in the first case \( T01 \). Otherwise, masses agree with cardinalities.

vertex and a marked edge incident upon it. Another point of view is that the rational numbers \( C_e/(2e) \) give the mass of planar trees with \( e \) edges, the mass of a planar tree \( \tau \) being as usual \( 1/|\text{Aut}(\tau)| \) with \( \text{Aut}(\tau) \) its group of symmetries. Our polar dessins with \( e \) edges are constructed from planar trees with \( e \) edges by distinguishing vertices and/or adding half-edges. Marking a vertex corresponds to multiplying by the number of vertices \( e + 1 \). Adding a half edge corresponds to multiplying by \( 2e \). Adjoining a second half edge corresponds to multiplying by \( 2e + 1 \). The total mass of the sets \( F_{m,n} \) is thus as given in Table 7.1.
8. Generic Monodromy

The main result of this section is that if a quasiChebyshev cover does not have automorphisms then its monodromy group is the full alternating or symmetric group on its degree. The fact that the proof goes through in the quasi setting indicates the naturality of this setting. To prove this monodromy result we proceed in stages. Irreducibility is obvious from the fact that \( \mathbb{P}_n^3 \) is connected. Primitivity follows from the fact that the ramification partitions associated to \( F_{mn} : \mathbb{P}_n^3(C) \to \mathbb{P}_n^3(C) \) do not allow for an intermediate curve, unless \( F_{mn} \) has automorphisms. Finally irreducibility is deduced from the fact that the monodromy group contains \( g_1 \), which has cycle structure \( mN^1_m \).

9. Field discriminants of specializations

One would ideally like to have explicit descriptions of \( p \)-adic ramification in the polynomials \( F_{mn}(s, x) \) for all \( F_{mn} \), all primes \( p \), and all \( s \in \mathbb{Q}_p^* \). Experimentation shows very regular behavior in all cases.

Let \( \mathbb{F}_p \) be an algebraic closure of \( \mathbb{F}_p \). Let \( \mathbb{Q}_p^m \) be the induced maximal unramified extension of \( \mathbb{Q}_p \). It is best to work geometrically, meaning factoring over \( \mathbb{Q}_p^m \), rather than \( \mathbb{Q}_p \). The dominant phenomenon is that \( F_{mn}(s, x) \) factors \( p \)-adically into factors which look very similar to each other with a few exceptions.

A clean example is provided by \( T_{8,9}(-1, x) \) with \( p = 3 \). Its polynomial discriminant at 3 is \( 3^C \) with \( C = 72 \). It factors over \( \mathbb{F}_3 \) as \( x^9(x + 1)^9(x^2 + 1)^9 \) and thus over \( \mathbb{F}_3 \) as \( x^9(x + 1)^9(x + i)^9(x - i)^9 \). Write the field discriminant at 3 as \( 3^C \). Then both \( C \) and \( c \) must be distributed somehow over the four roots, via \( C = \sum C_r \) and \( c = \sum c_r \). Necessarily \( C_r - c_r \) is non-negative and even. The simplest behavior that one could hope for is even distribution and no drop, so that the \( C_r \) and \( c_r \) are all \( 72/4 = 18 \). This is indeed what happens.

A more representative example is provided by \( T_{8,9}(-1, x) \) with \( p = 2 \). It factors over \( \mathbb{F}_2 \) as \( x^4(x^2 + x^2 + 1)^8 \), giving roots \( 0, r_1, r_2, r_3 \) over \( \overline{\mathbb{F}}_2 \). Also, because of the degree drop, \( \infty \) must be considered a root of multiplicity eight. Discriminant exponents \( (C_r, c_r) \) are \((35, 11)\) for the quartic root \( x = 0, (24, 24) \) for the octic factors corresponding to the \( r_i \), and \((31, 31)\) for \( x = \infty \). Here one should regard the \( r_i \) as behaving typically and 0 and \( \infty \) as both behaving specially.

The starting point for analysis in general is a factorization modulo \( p \), as follows.

If \( p^j \) exactly divides \( n \), define

\[
\begin{align*}
e(T, m, n, p) &= \begin{cases} 
\min(n/2, m2^{j-1}) & \text{if } p = 2, \\
\min(k/2, m(p^j - 1)/2) & \text{if } p > 2,
\end{cases} \\
e(U, m, n, p) &= \min(k, m(p^j - 1)).
\end{align*}
\]

Then for \( p^j \) exactly dividing \( m \) or \( n \) as indicated and for \( s \) reduced to \( \mathbb{F}_p \cup \{\infty\} \), one has congruences

\[
\begin{align*}
T_{mn}(s, x) &\equiv T_{m/p^j,n}(s, x)^{p^j(x + 2)^m/2}, \\
T_{mn}(s, x) &\equiv T_{m,n/p^j}(s, x)^{p^j(x + 2)^c(T,m,n,p)}, \\
U_{mn}(s, x) &\equiv U_{m/p^j,n}(s, x)^{p^j}, \\
U_{mn}(s, x) &\equiv U_{m,n/p^j}(s, x)^{p^j(x - 2)^c(U,m,n,p)},
\end{align*}
\]
modulo $p$.

When $s$ reduces into $\mathbb{F}_p^+$ rather than into $\{0, \infty\}$, the factorizations just displayed are particularly powerful. For then the bases on the right are generically separable, having at worst a double root when $s$ is a root of the relevant $d_{m,n}(s)$. The factor $f(x)$ of $F_{m,n}(s,x)$ over $\mathbb{Q}_p^{un}$ corresponding to a generic root is irreducible of degree $p^j$. It has polynomial discriminant $p^{j\gamma}$ and field discriminant also $p^{\gamma j}$. Moreover $K_j = \mathbb{Q}_p^{un}[x]/f(x)$ has an increasing chain of subfields $K_i$ of degree $p^i$ and field discriminant $p^{ij}$. Thus the slopes measuring wild ramification in the normalization of $\mathbb{Q}_p^{un}$ are $i + 1/(p-1)$ for $i = 1, \ldots, j$. This is less than the maximally wild case for extensions of $\mathbb{Q}_p^{un}$ of degree $p$, as there slopes are $i + 1 + 1/(p-1)$ for $i = 1, \ldots, j$. The source of exceptional factors are the special points $x = -2, 2, \infty$, and the roots of the relevant $U_{k'/2}$. Behavior of these factors needs to be described in a case-by-case way and often involves tame ramification.

When $s$ reduces into $\{0, \infty\}$, the $p$-adic factorization may involve larger degree factors. For example, $T_{8,9}(2, x)$ has an irreducible degree 32 factor over $\mathbb{Q}_2^{un}$. Also slopes of generic factors can reach the maximum of $j + 1 + 1/(p-1)$. For example, $T_{8,9}(3, x)$ factors over $\mathbb{Q}_3^{un}$ into four nonics, each having the maximum possible discriminant $3^{2n}$.

10. IMPRIMITIVE SPECIALIZATIONS

Consider a Chebyshev cover, thought of as a family of polynomials $F_{m,n}(s,x)$ with generic Galois group $G_N$ either $A_N$ or $S_N$. We know that for generic $\sigma \in \mathbb{Q}$, the Galois group of $F_{m,n}(\sigma,x)$ is all of $G_N$. However we are interested in constructing number fields by specializing at the “most special” points $\sigma$. So there is some concern that Galois groups will drop dramatically at these points.

In this section, we experimentally find that there is indeed such a drop in two instances. First, for $k = n - m$ odd, $U_{m,n}(1, x)$ has degree $m(n-2)$. Within range of computation it always is irreducible, but it has subfield of index two. Moreover, this subfield is defined by $T_{m,n}(1, x)$ which has generic Galois group. Second, for $k$ arbitrary now, $T_{m,n}((-1)^k, x)$ has either 0, 1, or 2 factors of $(x + 1)$ and the remaining part $T_{m,n}((-1)^k, x)$ has degree a multiple of three. Within range of computation it is always irreducible, but has a subfield of index three.

For $k = 1, 2$ we strengthen the obvious conjecture that this pattern continues by being explicit as to how roots are to be grouped. The second case concerns $T_{m,m+1}(-1, x)$ for general $m$ and $T_{m,m+2}(1, x)$ for $m$ odd. The parity distinction drops out, and counting $-1$ only once even if it has multiplicity two, there are always exactly $m(m-1)/2$ roots. The roots, as suggested by Figure 5.1 in the case $T_{8,9}$, form columns of heights 1 through $m-1$ in the case $k = 1$ and 0 through $m-1$ in the case $k = 2$. For $a$, $b$, $c$ positive integers summing to $m+1$, let $\alpha_{abc}$ be the $b^{th}$ root from the bottom or top in the column with $a$ roots; here one counts alternately from the bottom or top as one considers the columns from left to right. Define new complex numbers $\beta_{abc} = \alpha_{abc} + \alpha_{cab} + \alpha_{bca}$, excluding the central case $a = b = c$ if it is present. Then the conjecture is that the monic polynomial with roots $\beta_{abc}$ is in fact in $\mathbb{Q}[x]$. Our triangular indexing on the roots of $T_{8,9}(-1, x)$ is indicated in Figure 10.1. The corresponding degree twelve polynomial

$$f(x) = x^{12} - 36x^{10} - 48x^9 + 378x^8 + 864x^7 - 984x^6 - 4320x^5 - 3285x^4 + 192x^3 + 864$$
indeed has Galois group $S_{12}$. Our strengthening of the 2-imprimitivity conjecture likewise makes of triangular indices for the roots of $T_{m,n}(1,x)$ and two sets of triangular indices for the roots of $U_{m,n}(1,x)$. It is to be hoped that a proof of our imprimitivity conjectures would add insight to the nature of Chebyshev covers.

Besides the mysterious imprimitivity phenomenon, there are two obvious sources of Galois drop. First, if $\sigma$ is of the form $F_{m,n}(x_0)$ then certainly $F_{m,n}(\sigma, x)$ has $x_0$ as a root and so $G_\sigma \subseteq G_{N-1}$. Second, if the discriminant $D_{m,n}(s)$ is not a square in $\mathbb{Z}[s]$ but $D_{m,n}(\sigma)$ is a square in $\mathbb{Z}$ then certainly $G_\sigma \subseteq A_N$ while the generic Galois group is $S_N$. For very low $(m,n)$, there are other systematic sources of Galois drops because the curves governing the drop to a given group may have genus zero. For example, consider the quartic cover family $U_{2,3}(s, x) = (x-2)(x+2)^3 - s(x+1)^4$ with Galois group $S_4$. It has $T_{2,3}(s, y)$ as its resolvent cubic. So for $\sigma$ of the form $T_{2,3}(y) = y^3/((y-1)^2(y+2))$, the quartic $U_{2,3}(\sigma, x)$ has Galois group within the dihedral group $D_4$. In all but very small degrees, there seem to be no further Galois drops. This lack of Galois drops is what we want for the purposes of the next section.

Figure 10.1. Triangular labels on the roots of $T_{8,9}(-1,x)$. Each label is placed at the corresponding root, except that imaginary parts are independently scaled in each column for better visibility. Root $\alpha_{abc}$ has 9 - $a$ roots in its column, and is either $b^{th}$ from the top and $c^{th}$ from the bottom, or vice versa, depending on the parity of the column.

11. Exceptional number fields

Consider degree $N$ number fields with Galois group $A_N$ or $S_N$ and discriminant divisible only by primes in a given finite non-empty set $S$. In [?], the expected
number of such fields was discussed and in particular such a field was defined to be *exceptional* when \( N \) is larger than a certain number \( N(S) \). There is currently no general way to construct exceptional fields for any given \( S \), and it was conjectured in [?] that for each \( S \) there are only finitely many exceptional fields.

The construction of this paper gives exceptional fields for many \( S \). Moreover, the fields go substantially beyond the stringent demand \( N > N(S) \) in two ways. First, very simply, \( N \) can be very much greater than \( N(S) \). But second, as described in Section 9, the fields constructed here are ramified much more lightly than is allowed by their degree.

**Fields with** \( S = \{2, 3\} \). The case \( S = \{2, 3\} \) has been specifically pursued in the literature. The main previous paper is [3], where fields of degree up through 33 are constructed. The main technique in [3] is specializing three- and four-point covers, exactly as in this paper.

One has \( N(\{2, 3\}) = 62 \) and so the previous fields do not come close to being exceptional. Our family \( T_{8, 9} \) with generic degree 36 goes slightly beyond the previous 33 while our family \( U_{8, 9} \) with generic degree 64 goes slightly into the exceptional range.

Of course, it remains to confirm that specializations of \( T_{8, 9} \) and \( U_{8, 9} \) behave generically. As in [3], besides \( s = 1 \) we use the twenty-one specialization points

\[
{-8, -3, -2, -1, \frac{1}{2}, -\frac{1}{3}, -\frac{1}{8}, \frac{1}{9}, \frac{1}{4}, \frac{2}{3}, \frac{1}{2}, \frac{3}{9}, \frac{4}{3}, \frac{2}{1}, \frac{3}{8}, \frac{5}{9}, \frac{2}{5}, \frac{7}{2}, \frac{3}{2}, \frac{4}{9}, \frac{3}{2}, \frac{4}{9}, \frac{5}{2}, \frac{3}{2}}
\]

coming from the orbits of 2, 3, 4, and 9 under the action of \( S_5 \) permuting the three cusps \( s = 0, 1, \infty \).

As we saw in Section ??, \( T_{8, 9}(-1) \) is imprimitive with \( \mathbb{Q}[x]/T_{8, 9}(-1, x) \) containing a degree twelve \( S_{12} \) subfield. It is different from the 106 degree twelve fields found in [3]. According to Section 10, \( U_{8, 9}(1, x) \) is imprimitive with \( T_{8, 9}(1, x) \) defining the corresponding subfield. This is indeed the case, with \( T_{8, 9}(1, x) \) having Galois group \( S_{28} \). The absolute field discriminant is \( 2^{82}3^{53}41 \) which is smaller than the absolute field discriminants of the twenty-three degree 28 fields found in [3], the lowest discriminant there being \( 2^{82}3^{55} \).

The polynomial \( T_{8, 9}(2, x) \) factors as \( (x - 2)f_{35}(x) \) with \( f_{35}(x) \) having Galois group all of \( S_{35} \). The remaining nineteen points from (11.1) yield four \( A_{36} \) fields and fifteen \( S_{66} \) fields, all distinct. The polynomial \( U_{8, 9}(s, x) \) at the twenty-one \( s \) in (11.1) yields four \( A_{64} \) fields and seventeen \( S_{64} \) fields. Thus in summary, Galois groups behave completely generically, given the general expectations presented in Section 10.

**A field with** \( S = \{3, 5\} \). There are fewer possibilities for wild ramification at \( p \) in algebras of a given degree as \( p \) increases. For this reason \( N(p_1, \ldots, p_k) \) decays as one \( p_i \) increases and the others are fixed. This explains why the threshold for exceptionalness \( N(\{3, 5\}) = 38 \) is markedly less than the threshold \( N(\{2, 3\}) = 62 \).

Generally speaking, it is indeed harder to construct 3-5-number fields than 2-3 number fields by the method of three point covers, because the analog of (??) for any set of odd primes is empty.

Our source of an exceptional field with discriminant \( \pm3^a5^b \) is the cover \( T_{25, 27}(s, x) \) at the specialization point \( s = 1 \). The polynomial \( T_{25, 27}(1, x) \) has degree 300 and Section 10 says the corresponding field has a degree 100 subfield. We have confirmed that indeed it does have a degree 100 subfield, the defining polynomial with
roots \( \beta_{abc} \) being

\[
T_{25,27}^{\text{red}}(1, x) = x^{100} - 625x^{99} + 193,050x^{98} - 39,288,375x^{97} + \cdots
\]

The coefficients of \( x^j \) increase monotonically in size until the coefficient of \( x^{15} \) which has 83 digits; then they decrease monotonically with the constant term having 77 digits. Direct computation shows that its discriminant has the form

\[
\text{disc}_x(T_{25,27}^{\text{red}}(1, x)) = 3^{614}5^{500}(23 \cdot 137 \cdot 25471 \cdot 31482349 \cdot C)^2.
\]

Here \( C \approx 4.2 \times 10^{1006} \) is a non-prime having no prime factor \( < 10^{18} \). As usual it is easy to confirm genericity, by Jordan’s criterion that a transitive subgroup of \( S_N \) containing an element or prime order \( \ell \in (N/2, N - 3) \) must be \( A_N \) or \( S_N \). Here \( T_{25,27}^{\text{red}}(1, x) \) is irreducible but modulo 2 factors as 71 + 14 + 12 + 3; thus the Galois group is all of \( A_{100} \).

**Five large degree fields with** \( S = \{2, 5\} \). The threshold for exceptionalness in our final explicit case is \( N(\{2, 5\}) = 49 \). To construct candidates for exceptional fields, we use the covers \( T_{125,128}(s, x) \) and \( U_{125,128}(s, x) \). From our discriminant formulas, we know the corresponding discriminants have the form \( \pm 2^a5^b(s - 1)^s d^2_3(s) \) with \( d^2_3(s) = s + 1 \) and \( d^2_5(s) = s + 27 \).

Special points which give algebras with discriminants of the form \( \pm 2^a5^b \) are \( s = -1, 1, 4/5, \) and \( 5/4 \) for \( T_{125,128}(s, x) \) and \( s = 5 \) for \( U_{125,128}(s, x) \). The degrees of these algebras are 7998, 7875, 8000, 8000, and 15875. The points \( s = 5/4, s = 4/5 \) introduce factors of \( 3^2 \) into the polynomial discriminant but these factors necessarily drop out in the field discriminant because the 3-adic proximity \( \text{ord}_3(s + 1) = 2 \) is a multiple of the ramification index \( e = 2 \).

As a convenient simpler parallel case, we can replace \((125, 128)\) by \((5, 8)\) and use the same specialization points. Then from \( T_{5,8}(s, x) \) at \( s = -1, 1, 4/5, \) and \( 5/4 \) we get fields of degree 18, 15, 20, 20. The first has a degree six subfield with Galois group \( S_6 \) and the last three have Galois groups \( S_{15}, S_{20}, S_{20}, \) all as expected. The field discriminants are respectively \( 2^{43}5^{17}, 2^{38}5^{15}, 2^{59}5^{36}, \) and \( 2^{59}5^{37} \). Similarly for \( U_{5,8}(s, x) \) at \( s = 5 \) we get a field of degree 35, field discriminant \( -2^{59}5^{67} \), and Galois group all of \( S_{35} \).

Our treatment of specialization issues has been extensive enough to give considerable confidence in our expectations about Galois groups. Thus \( T_{125,128}(-1, x)/(x + 1)^2 \) should be 3-imprimitive, with the subfield of degree 2666 should have Galois group all of \( S_{2666} \). The remaining four polynomials should define fields with Galois group the full symmetric group of their degree. Factorization over \( \mathbf{F}_p[x] \) to confirm our expectations is not computationally feasible. However information obtained by counting roots in \( \mathbf{F}_p \) statistically agrees with our Galois group expectations, increasing confidence in these specific cases.

Explicitly, our degree 15875 polynomial is

\[
U_{125,128}(5, x) = (x - 2)^3 u_{62,5}(x)^{128} - 5(x + 2)^{125} u_{64}(x)^{250}.
\]

Its existence depends not only on the general theory of Chebyshev covers but also on the two \( abc \)-triples \( 5^3 + 3 = 2^7 \) and \( 3^3 + 5 = 2^5 \). Remarkably, despite the presence of the prime 3 in both these triples, not even the polynomial discriminant of \( U_{125,128}(1, x) \) is divisible by 3.
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