A new result in game theory
Dave Roberts


**Example.** The distribution of support sizes $k$ in Cauchy-random zero-sum 50-by-70 matrix games. The most common support sizes are $k = 29$ and $k = 30$, with frequencies 14.8% and 14.6% respectively.
We work always with zero-sum games between two players.

**Familiar Example of Rock-Paper-Scissors.**
“Rock smashes scissors, scissors cuts paper, paper suffocates rock.” Coded up in a matrix:

<table>
<thead>
<tr>
<th></th>
<th>Rock</th>
<th>Paper</th>
<th>Scissors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rock</td>
<td>0</td>
<td>−1</td>
<td>1</td>
</tr>
<tr>
<td>Paper</td>
<td>1</td>
<td>0</td>
<td>−1</td>
</tr>
<tr>
<td>Scissors</td>
<td>−1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Kids know that best play is to randomly mix the strategies. In code, row’s best strategy is

\[ x = (x_1, x_2, x_3) = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right). \]

Column’s best strategy is similarly

\[ y = (y_1, y_2, y_3) = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right). \]

This is a fair game, in the sense that on average the column player pays the row player \( \lambda = 0 \).
Example of biased Rock-Paper-Scissors.

<table>
<thead>
<tr>
<th>Row Player</th>
<th>Column Player</th>
<th>Rock</th>
<th>Paper</th>
<th>Scissors</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Rock</td>
<td>0</td>
<td>−1</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Paper</td>
<td>1</td>
<td>c</td>
<td>−1</td>
</tr>
<tr>
<td></td>
<td>Scissors</td>
<td>−1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Unique solution if $0 \leq c < 3$:

\[
(x_1, x_2, x_3) = \left(\frac{1}{3} + \frac{c}{9}, \frac{1}{3}, \frac{1}{3} - \frac{c}{9}\right)
\]

\[
(y_1, y_2, y_3) = \left(\frac{1}{3} - \frac{c}{9}, \frac{1}{3}, \frac{1}{3} + \frac{c}{9}\right)
\]

\[
\lambda = \frac{c}{9}
\]

(Each player still mixes all three strategies.)

Unique solution if $c > 3$:

\[
(x_1, x_2, x_3) = \left(\frac{2}{3}, \frac{1}{3}, 0\right)
\]

\[
(y_1, y_2, y_3) = \left(\frac{2}{3}, 0, \frac{1}{3}\right)
\]

\[
\lambda = \frac{1}{3}
\]

(Each player now uses only two strategies.)
General set-up. A game is given by an $m$-by-$n$ matrix $A = (a_{ij})$.

The row player can choose to mix his strategies according to any probability vector $x = (x_i)$.

The column player can choose to mix his strategies according to any probability vector $y = (y_j)$.

Theorem (von Neumann (1928)) For “almost all” games, there is a unique best strategy mix $x$ for the row player and a unique best strategy mix $y$ for the column player.

Theorem (Shapley & Snow (1950)) In their mixed best strategies, both players mix the same number $k$ of their pure strategies.

The number $k$ is the support size of the game.
A typical 5-by-7 game:

\[-18 \quad -54 \quad -70 \quad -93 \quad 40 \quad -25 \quad -64\]
\[-1 \quad 93 \quad 49 \quad 141 \quad -116 \quad -417 \quad 89\]
\[173 \quad 5 \quad -347 \quad -2716 \quad 254 \quad 83 \quad 58\]
\[-652 \quad -84 \quad -66 \quad -119 \quad 76 \quad 1212 \quad -591\]
\[33 \quad -80 \quad 53 \quad 18 \quad 219 \quad 47 \quad 196\]

Its solution:

\[k = 4\]
\[x \approx (0, 32\%, 3\%, 7\%, 58\%)\]
\[y \approx (29\%, 50\%, 0, 3\%, 0, 18\%, 0)\]
\[\lambda \approx -21.93\]

The game and its solution again:

<table>
<thead>
<tr>
<th>29% 50%</th>
<th>3% 18%</th>
</tr>
</thead>
<tbody>
<tr>
<td>-18 -54 -70 -93 40 -25 -64</td>
<td></td>
</tr>
<tr>
<td>-1 93 49 141 -116 -417 89</td>
<td></td>
</tr>
<tr>
<td>173 5 -347 -2716 254 83 58</td>
<td></td>
</tr>
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<td>-652 -84 -66 -119 76 1212 -591</td>
<td></td>
</tr>
<tr>
<td>33 -80 53 18 219 47 196</td>
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**Natural Question:** How is the support size $k$ distributed for “random” $m$-by-$n$ games?

Progress was made by

- Goldman 1957 $k=1$
- Thrall & Falk 1965 Small $m$, $n$
- Faris & Maier 1987 Small $m$, $n$
- Berg & Engel 1998 Asymptotics

Goldman’s result was that, for any reasonable notion of randomness, $k = 1$ occurs with probability $\frac{m!n!}{(m+n-1)!}$. For large $m$ and $n$, Goldman’s quantity becomes very small.

Berg & Engel used remarkable statistical mechanics techniques to conjecture a formula valid for matrices with any given shape but only in the limit of large size.
**Experimental data.** I took experimental data in the spirit of Thrall & Falk and Faris & Maier. For example, from 1,000,000 different 5-by-7 games chosen with $a_{ij}$ independent and normally distributed, I found support sizes distributed as

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.46%</td>
<td>18.26%</td>
<td>45.58%</td>
<td>30.22%</td>
<td>4.48%</td>
</tr>
</tbody>
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1.46% compares well with Goldman’s proved $1/66 \approx 1.51\%$.

**Guessing a formula.** I multiplied all the above percentages by 66, obtaining

<table>
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<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.96</td>
<td>12.05</td>
<td>30.08</td>
<td>19.95</td>
<td>2.96</td>
</tr>
</tbody>
</table>

These are all within experimental error of being integers. The same phenomenon occurred for all other small $(m, n)$. From the rational numbers obtained, I conjectured a general Goldman-style formula. So, e.g., my formula predicted that a random 5-by-7 game would have support size four with probability $20/66 = 30.30\%$. 

Finding the right notion of randomness. My conjectured formula seemed roughly right for many notions of randomness. But it was likely to be exactly right for at most one notion of randomness. Looking at Goldman’s proof led me to choose the $a_{ij}$ independently with respect to the Cauchy density $\frac{1}{\pi(x^2+1)}$ rather than the Gaussian density $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$.

The two densities are qualitatively similar but the Cauchy density has fatter tails.
Main result. With the right density, I succeeded in establishing my conjectured formula:

**Theorem** For Cauchy-random $m$-by-$n$ zero-sum matrix games, the chance that the support size is $k$ is given by

\[
\frac{m!}{k!(m-k)!} \frac{n!}{k!(n-k)!} \frac{(m-1)!(n-1)!}{(m+n-1)!} k.
\]

**Proof.** The proof is a complicated induction. It involves reducing multivariate integrals to univariate integrals by using special properties of the Cauchy density.

**Very simple concluding principle.** The theorem implies that *The support size $k$ is approximately normally distributed with mean $\frac{mn}{m+n}$ and standard deviation $\frac{mn}{(m+n)^{3/2}}$.***