Motivic Computations in Magma
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1. Invariants of algebraic varieties

2. Familiar examples

3. Hypergeometric motives

4. L-factors of Hypergeometric motives at wild primes, an example
1. **Invariants of algebraic varieties.** Let $X$ be a smooth projective variety over $\mathbb{Q}$ of dimension $d$. The $2d$-dimensional real manifold $X(\mathbb{C})$ has cohomology groups $H^w(X(\mathbb{C}), \mathbb{Q})$. Because of the arithmetic source, these cohomology groups have extra structure.

Three types of extra structures which are local in the sense that they are associated with a place $v \in \{\infty, 2, 3, 5, 7, \ldots\}$ of $\mathbb{Q}$:

**From the infinite place** $v = \infty$. There is a Hodge decomposition

$$H^w(X(\mathbb{C}), \mathbb{C}) = \bigoplus_{p=0}^{w} H^{p,w-p}.$$ 

with $H^{p,q} = \overline{H^{q,p}}$. In particular the $w^{th}$ Betti number decomposes $b = \sum h^{p,w-p}$ with

$$h^{p,q} = h^{q,p}.$$ 

Complex conjugation $\text{Fr}_{\infty} : X(\mathbb{C}) \to X(\mathbb{C})$ interchanges $H^{p,q}$ and $H^{q,p}$ so that moreover

$$h_{w/2,w/2} = h_{+}^{w/2,w/2} + h_{-}^{w/2,w/2} \quad (\text{for } w \text{ even}).$$
From a good place $p$. From the integers $|X(\mathbb{F}_{p^f})|$, one gets a Frobenius operator $\text{Fr}_p$ on $H^w(X(\mathbb{C}), \mathbb{Q})$, well-defined up to conjugation (Dwork, Grothendieck, Deligne). The eigenvalues $\alpha_i$ have absolute value $p^w/2$. It is best to package them into a polynomial

$$F_p(H^w(X(\mathbb{C}), \mathbb{Q}), T) = \prod_{i=1}^{b} (1 - \alpha_i T) \in \mathbb{Z}[T].$$

From a bad place $p$. The case is similar to the case of good places, but much more complicated, even still depending on resolution-of-singularities type conjectures for some basic statements. One still has a polynomial $F_p(H^w(X(\mathbb{C}), \mathbb{Q}), T)$, now of degree $< b$. One also has a conductor $p^{np}$, with $n_p \in \mathbb{Z}_{\geq 1}$ measuring the badness of the mod $p$ reduction, from a cohomological viewpoint.
**L-functions.** The local invariants discussed so far are naturally combined into a global L-function

\[
L(H^w(X(\mathbb{C}), \mathbb{Q}), s) = \Gamma(h, s) \prod_p \frac{1}{F_p(H^w(X(\mathbb{C}), \mathbb{Q}), p^{-s})}.
\]

This function conjecturally satisfies a functional equation \( s \leftrightarrow w + 1 - s \) in which the conductor \( N = \prod_p p^{n_p} \) enters fundamentally and measures computational complexity.

One can also work with suitable direct summands \( M \subseteq H^w(X(\mathbb{C}), \mathbb{Q}) \) called motives, where all the above invariants are defined. The motivic L-functions \( L(M, s) \) are very well implemented in *Magma* (Dokchitser, Watkins). They are the subject of many conjectures: connections to automorphic forms, special values at integers, Riemann hypothesis and further properties of zeros, . . . .

A natural problem is to produce motives \( M \) with small conductor \( N \) for given Hodge numbers \( h \) (and given “motivic Galois group”)


2. **Familiar examples.** Our examples are in increasing order of \( w = \dim(X) \) and focus on \( M \subseteq H^w(X(\mathbb{C}), \mathbb{Q}) \).

**A. Number Fields.** Taking \( X \) of dimension zero, the study of \( M = H^0(X(\mathbb{C}), \mathbb{Q}) \) is the study of number fields \( K \) in a fancy language. If \( K \otimes \mathbb{R} = \mathbb{R}^r \times \mathbb{C}^s \) then signature translates to Hodge numbers:

\[
(r, s) = (h^0_+, h^0_-, 2h^0_-).
\]

The absolute discriminant translates to conductor: \( D = N \). The Dedekind zeta function is an example of an \( L \)-function: \( \zeta(K, s) = L(M, s) \).

**B. Curves.** Taking \( X \) of dimension one, the study of \( M = H^1(X(\mathbb{C}), \mathbb{Q}) \) is essentially the study of Jacobians in a fancy language. An example of *Magma* functionality:

```magma
gtT := PolynomialRing(Integers());
gP2<x,y,z> := ProjectiveSpace(FiniteField(5),2);
gFactorization(gPolynomial(gCurve(gP2,x^4+y^4+z^4)));
[<5*T^2 - 2*T + 1, 3>]
```
C. Surfaces and their Hodge numbers.

\[ \text{P3}<w,x,y,z> := \text{ProjectiveSpace}(\text{Rationals}(),3) ; \]
\[ \text{fermat} := \text{func}<n|\text{Surface}(\text{P3},w^n+x^n+y^n+z^n)> ; \]
\[ \text{hodgesquare} := \text{func}<S| \]
\[ \text{HodgeNumber}(S,i,j):i \text{ in } [0..2]] : \]
\[ j \text{ in } [2..0 \text{ by } -1] > ; \]
\[ \text{hodgesquare}(\text{fermat}(n)) : n \text{ in } [1..5]] ; \]

\[
\begin{bmatrix}
0, 0, 1 \\
0, 1, 0 \\
1, 0, 0
\end{bmatrix}, \quad \text{The projective plane } \mathbb{P}^2
\]

\[
\begin{bmatrix}
0, 0, 1 \\
0, 2, 0 \\
1, 0, 0
\end{bmatrix}, \quad \text{The quadric } \mathbb{P}^1 \times \mathbb{P}^1
\]

\[
\begin{bmatrix}
0, 0, 1 \\
0, 7, 0 \\
1, 0, 0
\end{bmatrix}, \quad \mathbb{P}^2 \text{ with six points blown up}
\]

\[
\begin{bmatrix}
1, 0, 1 \\
0, 20,0 \\
1, 0, 1
\end{bmatrix}, \quad \text{K3 surface}
\]

\[
\begin{bmatrix}
4, 0, 1 \\
0, 45,0 \\
1, 0, 4
\end{bmatrix}, \quad \text{General type surface}
\]
3. Hypergeometric Motives. Let $d$ be an integer. Let $\alpha_1, \ldots, \alpha_d, \beta_1, \ldots, \beta_d \in \mathbb{Q}/\mathbb{Z}$ with always $\alpha_i \neq \beta_j$. Suppose the number of times a rational number occurs depends only on its denominator. Then for every $t \in \mathbb{Q} - \{0, 1\}$ one has a rank $d$ motive

$$M_t = H(\alpha_1, \ldots, \alpha_d; \beta_1, \ldots, \beta_d; t).$$

One would like to compute the L-series $L(M_t, s)$ completely.

The Hodge numbers depend only on how the $\alpha$’s and the $\beta$’s intertwine in the circle $\mathbb{R}/\mathbb{Z}$. At one extreme, if the $\alpha$’s and the $\beta$’s are separated, then

$$(h^{d-1, 0}, \ldots, h^{0, d-1}) = (1, 1, \ldots, 1, 1).$$

At the other extreme,

$$(h^{0, 0}) = (d)$$

if there is complete intertwining.
Magma’s HGM package (Watkins) implements much of what is known about HGMs:

- It computes Hodge numbers and their parity split, hence \( L_{\infty}(M, s) = \Gamma(h, s) \).

- For \( p \) not wild, meaning not dividing a denominator of an \( \alpha_i \) or a \( \beta_j \), it very efficiently computes (Katz, Cohen, \ldots) the L-factor \( L_p(M, s) = 1/F_p(M, p^{-s}) \) and the conductor \( p^{np} \).

- For \((\alpha, \beta) \) “sufficiently classical,” it identifies \( M \) as coming from a specific variety \( X \) and sometimes thereby computes \( L_p(M, s) \) for wild primes \( p \) too.

The HGM package feeds nicely into the L-Series package, but the ambiguity at wild primes limits its usefulness.
4. Wild L-factors of HGMs. Example 1 (works perfectly!):

\[ M1 := \text{HypergeometricData}([1/6,1/3,2/3,5/6], \]
\[ [0,1/4,1/2,3/4]); \]
\[ L1 := \text{LSeries}(M1,2); \quad (t = 2 \text{ as an example}) \]
\[ \text{EulerFactor}(L1,2); \]
\[ 1-T \]
\[ \text{EulerFactor}(L1,3); \]
\[ 1 \]
\[ \text{Factorization}((\text{Conductor}(L1))); \]
\[ [ <2, 10>, <3, 6> ] \]
\[ \text{CheckFunctionalEquation}(L1); \]
\[ 1.31266454628776364420511633263E-27 \approx 0 \checkmark \]

Here the wild primes are 2 and 3. The \( \alpha_i \)'s and \( \beta_j \)'s completely intertwine, so the Hodge vector is \( (h^{0,0}) = (4) \). \textit{Magma} identifies

\[ L(M1,t,s) = \frac{\zeta(\mathbb{Q}[x]/(729x^2(x - 1)^4t - 16),s)}{\zeta(\mathbb{Q}[x]/(x^2 - t),s)}. \]

Because of this identification, the local factors \( L_p(M1_2,s) \) and local conductors \( p^{np} \) have been correctly calculated.
Example 2 (problem at 3):

```plaintext
> M2 := HypergeometricData([1/3,1/3,2/3,2/3],[0,0,0,0]);
> L2 := LSeries(M2,2); (t = 2 as an example)
WARNING: Guessing wild prime information
> EulerFactor(L2,2);
1 (2 is tame, so right)
> EulerFactor(L2,3);
1 (3 is wild, so dubious (but right))
> Factorization(Conductor(L2));
[<2, 4>] (right at 2, dubious at 3 (and wrong))
> CheckFunctionalEquation(L2);
|Sign| is nowhere near 1,
wrong functional equation?
-0.3074980512625301093947618264671 ≠ 0  X
```

Here the only wild prime is 3. The $\alpha_i$'s and $\beta_j$'s are completely separated, so the Hodge vector is $(h^{3,0}, h^{2,1}, h^{1,2}, h^{0,3}) = (1, 1, 1, 1)$. Any underlying variety has to have dimension at least three. *Magma* cannot appeal to its more classical parts to get the correct invariants at 3.
Connection between the two examples. An element of \( \mathbb{Q}/\mathbb{Z} \) is the sum of its \( p \)-primary parts. E.g. \( 1/6 = 1/2 + 2/3 \) has 2-primary part \( 1/2 \) and 3-primary part \( 2/3 \). In our case, taking 3-primary parts of the indices of

\[
M_1 = H([1/6, 1/3, 2/3, 5/6], [0, 1/4, 1/2, 3/4])
\]
gives the indices of

\[
M_2 = H([1/3, 1/3, 2/3, 2/3], [0, 0, 0, 0]).
\]

Via this connection, as an instance of a general theorem, one has the following fact: If \( \text{ord}_3(N_1t) \geq 6 \) then \( F_3(M_1t, T) = F_3(M_2t, T) = 1 \) and \( c_3(M_1) = c_3(M_2) \).

In other words, using the ability of the L-series package to input specified bad factors,

\[
\text{LSeries}(M_2, t : \text{BadPrimes} := [<3, \text{Valuation(Conductor(LSeries(M_1, t)))}, 3), 1>])
\]
gives the right L-series when \( \text{ord}_3(N_1t) \geq 6 \).
As numerical examples, all double-checked with CheckFunctionalEquation, some conductors are as follows.

\[
\begin{array}{c|cc|cc}
  t & N1_t & N2_t & N1_t & N2_t \\
  \hline
  -1 & 2^9 \times 3^6 & 2^1 \times 3^6 \\
  1/2 & 2^{11} \times 3^6 & 2^3 \times 3^6 \\
  2 & 2^{10} \times 3^6 & 2^4 \times 3^6 & \leftarrow \text{was 0, now fixed!} \\
  -8 & 2^8 \times 3^6 & 2^2 \times 3^6 \\
  -1/8 & 2^{11} \times 3^6 & 2^3 \times 3^6 \\
  -2 & 2^{10} \times 3^7 & 2^4 \times 3^7 \\
  8/9 & 2^8 \times 3^8 & 2^2 \times 3^8 \\
  -1/3 & 2^6 \times 3^9 & 2^1 \times 3^9 \\
  1/3 & 2^9 \times 3^9 & 2^1 \times 3^9 \\
  2/3 & 2^{10} \times 3^9 & 2^4 \times 3^9 \\
  4/3 & 2^6 \times 3^9 & 2^4 \times 3^9 \\
  9/8 & 2^{11} \times 3^{10} & 2^3 \times 3^{10} \\
  -3 & 2^6 \times 3^{10} & 2^1 \times 3^{10} \\
  3/4 & 2^4 \times 3^{10} & 2^3 \times 3^{10} \\
  3/2 & 2^{11} \times 3^{10} & 2^3 \times 3^{10} \\
  3 & 2^9 \times 3^{10} & 2^1 \times 3^{10} \\
\end{array}
\]

We hope to implement the general theorem so as to allow Magma to compute most low degree Hypergeometric L-series completely.