Questions

Example Is \( \sum_{n=1}^{\infty} \frac{\ln n}{n} \) convergent or divergent?

Example Is \( \sum_{n=1}^{\infty} \frac{2n^2 + 7n}{3^n(n^2 + 5n - 1)} \) convergent or divergent?

Example Suppose \( \sum a_n \) and \( \sum b_n \) are series with positive terms and \( \sum b_n \) is known to be convergent.

(a) If \( a_n > b_n \) for all \( n \), what can you say about \( \sum a_n \)? Why?
(b) If \( a_n < b_n \) for all \( n \), what can you say about \( \sum a_n \)? Why?

Example Determine whether the series converges or diverges. \( \sum_{n=1}^{\infty} \frac{1}{n^2 + n + 1} \).

Example Is \( \sum_{n=1}^{\infty} \frac{n}{2^n(n + 1)} \) convergent or divergent?

Example Determine whether the series converges or diverges. \( \sum_{n=1}^{\infty} \frac{1}{1 + \sqrt{n}} \).

Example Suppose that \( \sum a_n \) and \( \sum b_n \) are series with positive terms and \( \sum b_n \) is convergent. Prove that if
\[
\lim_{n \to \infty} \frac{a_n}{b_n} = 0
\]
then \( \sum a_n \) is convergent.

Note: This is a more difficult problem than most, since it is a proof that involves the definition of limit.

Example Suppose that \( \sum a_n \) and \( \sum b_n \) are series with positive terms and \( \sum b_n \) is divergent. Prove that if
\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \infty
\]
then \( \sum a_n \) is divergent.

Note: This is a more complicated problem than most, and involves using a proof by contradiction.

Solutions

Example Is \( \sum_{n=1}^{\infty} \frac{\ln n}{n} \) convergent or divergent?

Let’s try to use the Comparison Test. How do we know what series to compare to? Well, we try something, and use a series which we know something about. We usually try to pick our comparison series based on attributes of the given
series.

\[
\ln n > 1 \quad \text{for } n \geq 3
\]
\[
a_n = \frac{\ln n}{n} > \frac{1}{n} = b_n \quad \text{for } n \geq 3
\]

Since \(\sum_{n=3}^{\infty} \frac{1}{n}\) is divergent (it is a p-series with \(p = 1\)), we know \(\sum_{n=1}^{\infty} \frac{\ln n}{n}\) is divergent by the comparison test.

**Example** Is \(\sum_{n=1}^{\infty} \frac{2n^2 + 7n}{3^n(n^2 + 5n - 1)}\) convergent or divergent?

Let’s try to use the Limit Comparison Test. For large \(n\),

\[
2n^2 + 7n \sim 2n^2
\]
\[
3^n(n^2 + 5n - 1) \sim 3^n n^2
\]

So let’s take

\[
a_n = \frac{2n^2 + 7n}{3^n(n^2 + 5n - 1)} \quad b_n = \frac{2n^2}{3^n n^2} = \frac{2}{3^n}
\]

The series \(\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{2}{3} \left(\frac{1}{3}\right)^{n-1}\) is a convergent geometric series (since \(a = 2/3, |r| = 1/3 < 1\)).

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \left(\frac{\frac{2n^2 + 7n}{3^n(n^2 + 5n - 1)}}{\frac{2}{3^n}}\right)
\]
\[
= \lim_{n \to \infty} \left(\frac{2n^2 + 7n}{3^n(n^2 + 5n - 1)} \cdot \frac{3^n}{2}\right)
\]
\[
= \lim_{n \to \infty} \left(\frac{2n^2 + 7n}{2(n^2 + 5n - 1) + 1/n^2}\right)
\]
\[
= \lim_{n \to \infty} \left(\frac{2 + 7/n}{2 + 1/n + 5/n - 1/n^2}\right)
\]
\[
= 1 > 0 \text{ and finite.}
\]

Since \(\sum b_n\) converges, \(\sum a_n\) converges by the limit comparison test.

**Example** Suppose \(\sum a_n\) and \(\sum b_n\) are series with positive terms and \(\sum b_n\) is known to be convergent.

(a) If \(a_n > b_n\) for all \(n\), what can you say about \(\sum a_n\)? Why?
(b) If \(a_n < b_n\) for all \(n\), what can you say about \(\sum a_n\)? Why?
(a) If \( a_n > b_n \) for all \( n \), and \( \sum b_n \) is convergent, then we cannot say anything about \( \sum a_n \) since it is not bounded above by \( \sum b_n \).

(b) Since \( a_n \) is positive, the series \( \sum a_n \) must be increasing since we are always adding a positive quantity to the partial sum (in other words, \( s_{n+1} > s_n \)). If \( a_n < b_n \) for all \( n \), and \( \sum b_n \) is convergent, then \( \sum a_n \) is convergent since it is bounded above by \( \sum b_n \) which converges.

**Example (11.4.3)** Determine whether the series converges or diverges. \[ \sum_{n=1}^{\infty} \frac{1}{n^2 + n + 1}. \]

Let’s try to use the Comparison Test. Let’s try to pick our comparison series based on attributes of the given series.

\[ a_n = \frac{n^2 + n + 1}{n^2 + n + 1} > \frac{n^2}{n^2} \text{ for } n \geq 1 \]

\[ a_n = \frac{1}{n^2 + n + 1} < \frac{1}{n^2} = b_n \text{ for } n \geq 1 \text{ (note change in the relation)} \]

Since \( \sum b_n = \sum_{n=1}^{\infty} \frac{1}{n^2} \) is convergent (it is a \( p \)-series with \( p = 2 \)), we know \( \sum a_n = \sum_{n=1}^{\infty} \frac{1}{n^2 + n + 1} \) is convergent by the comparison test.

**Example** Is \( \sum_{n=1}^{\infty} \frac{n}{2^n(n+1)} \) convergent or divergent?

Let’s try to use the Limit Comparison Test. For large \( n \),

\[ 2^n(n+1) \sim 2^n n \]

So let’s take

\[ a_n = \frac{n}{2^n(n+1)} \quad b_n = \frac{n}{2^n} = \frac{1}{2^n} \]

The series \( \sum b_n = \sum_{n=1}^{\infty} \frac{1}{2} \left( \frac{1}{2} \right)^{n-1} \) is a convergent geometric series (since \( a = 1/2, \ |r| = 1/2 < 1 \)).

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \left( \frac{\frac{n}{2^n(n+1)}}{\frac{1}{2^n}} \right) = \lim_{n \to \infty} \left( \frac{n}{2^n(n+1)} \right) (2^n) = \lim_{n \to \infty} \left( \frac{n}{n+1} \right) = \lim_{n \to \infty} \left( \frac{1}{1 + 1/n} \right) = 1 > 0 \text{ and finite.} \]
Since $\sum b_n$ converges, $\sum a_n$ converges by the limit comparison test.

**Example** Determine whether the series converges or diverges. $\sum_{n=1}^{\infty} \frac{1}{1 + \sqrt{n}}$.

Let’s try to use the Comparison Test. Let’s try to pick our comparison series based on attributes of the given series.

$a_n = \frac{1 + \sqrt{n}}{1 + \sqrt{n}} < \frac{1}{\sqrt{n}} = b_n$ for $n \geq 1$ (note change in the relation)

Since $\sum b_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent (it is a $p$-series with $p = 1/2$), this doesn’t tell us anything about $\sum a_n$ (see (11.1.1)).

Since this doesn’t help us, we’ll have to try something else.

Let’s try the limit comparison test with the comparison series $\sum b_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{\left(\frac{1}{1 + \sqrt{n}}\right)}{\left(\frac{1}{\sqrt{n}}\right)} = \lim_{n \to \infty} \left(\frac{1}{1 + \sqrt{n}}\right) \left(\sqrt{n}\right) = \lim_{n \to \infty} \left(\frac{\sqrt{n}}{1 + \sqrt{n}}\right) = \lim_{n \to \infty} \left(\frac{1}{1 + 1/\sqrt{n}}\right) = 1 > 0$$

Since $\sum b_n$ diverges, $\sum a_n$ diverges by the limit comparison test.

**Example** Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms and $\sum b_n$ is convergent. Prove that if

$$\lim_{n \to \infty} \frac{a_n}{b_n} = 0$$

then $\sum a_n$ is convergent.

Note: This is a more difficult problem than most, since it is a proof that involves the definition of limit.

Since $\lim_{n \to \infty} \frac{a_n}{b_n} = 0$, by the definition of limit we know there exists an $N > 0$ such that $|a_n/b_n - 0| < 1$ for all $n > N$.

Since $a_n$ and $b_n$ are positive, $|a_n/b_n - 0| < 1 \implies a_n < b_n$.

Therefore, since $\sum b_n$ converges, $\sum a_n$ converges by the comparison test.
Example Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms and $\sum b_n$ is divergent. Prove that if
\[ \lim_{n \to \infty} \frac{a_n}{b_n} = \infty \]
then $\sum a_n$ is divergent.

Note: This is a more complicated problem than most, and involves using a *proof by contradiction*.

Assume $\sum a_n$ converges.

Since $\lim_{n \to \infty} \frac{a_n}{b_n} = \infty \implies \lim_{n \to \infty} \frac{b_n}{a_n} = 0$.

Using the result from Problem (11.4.40 a), we know that if $\sum a_n$ converges then $\sum b_n$ converges as well.

But we are told that $\sum b_n$ diverges (contradiction). Therefore, the assumption we made must be wrong, and $\sum a_n$ diverges.