Questions

Example Find the absolute maximum and absolute minimum values of \( f(x) = \frac{x}{x^2 + 1} \) on the interval \([0, 2]\).

Example Find the absolute maximum and absolute minimum values of \( f(x) = \frac{\ln x}{x} \) on the interval \([1, 3]\).

Example If \( a \) and \( b \) are positive numbers, find the maximum value of \( f(x) = x^a(1-x)^b \), \( 0 \leq x \leq 1 \).

Solutions

Example Find the absolute maximum and absolute minimum values of \( f(x) = \frac{x}{x^2 + 1} \) on the interval \([0, 2]\).

Absolute extrema on a closed interval are found using the Closed Interval Method:
1) Find the values of \( f \) at the critical numbers of \( f \) in \((a, b)\).
2) Find the values of \( f \) at the endpoints of the interval.
3) The largest of the values from 1) and 2) is the absolute maximum; the smallest of these values is the absolute minimum.

We need the critical numbers. We need to find where \( f'(c) = 0 \) and where \( f'(x) \) does not exist. Since \( x^2 + 1 \neq 0 \) for real valued \( x \), the derivative always exists.

\[
f(x) = \frac{x}{x^2 + 1}
\]
\[
f'(x) = \frac{d}{dx} \left[ \frac{x}{x^2 + 1} \right] = \frac{(x^2 + 1) \frac{d}{dx} [x] - x \frac{d}{dx} [x^2 + 1]}{(x^2 + 1)^2}
\]
\[
= \frac{(x^2 + 1)(1) - x(2x)}{(x^2 + 1)^2}
\]
\[
= \frac{x^2 + 1 - 2x^2}{(x^2 + 1)^2}
\]
\[
= \frac{-x^2 + 1}{(x^2 + 1)^2}
\]

For \( f'(c) = 0 \), the numerator must equal zero,

\[
f'(c) = 0 = \frac{-c^2 + 1}{(c^2 + 1)^2}
\]
\[
0 = -c^2 + 1
\]
\[
c^2 = 1
\]
\[
c = \pm 1
\]

We have shown that \( f'(1) = 0 \) and \( f'(-1) = 0 \). The critical numbers are \(+1, -1\). Only \(+1\) is in the interval \((0, 2)\).
Now, we evaluate the function at the critical numbers in the interval and at the endpoints of the interval:

\[
\begin{align*}
 f(1) &= \frac{1}{(1)^2 + 1} = \frac{1}{2} \\
 f(0) &= \frac{0}{(0)^2 + 1} = 0 \\
 f(2) &= \frac{2}{(2)^2 + 1} = \frac{2}{5}
\end{align*}
\]

The largest number is \(1/2\), so this is the absolute max and it occurs at \(x = +1\). The smallest number is 0, so this is the absolute min and it occurs at \(x = 0\).

**Example** Find the absolute maximum and absolute minimum values of \(f(x) = \frac{\ln x}{x}\) on the interval \([1, 3]\)

Absolute extrema on a closed interval are found using the Closed Interval Method:
1) Find the values of \(f\) at the critical numbers of \(f\) in \((a, b)\).
2) Find the values of \(f\) at the endpoints of the interval.
3) The largest of the values from 1) and 2) is the absolute maximum; the smallest of these values is the absolute minimum.

We need the critical numbers, which means we need to find where \(f'(c) = 0\) and where \(f'(x)\) does not exist. The only place we could have the derivative not defined is for \(x \leq 0\); luckily, this is outside of the interval \((1, 3)\) so we don’t need to worry about the derivative being undefined.

\[
\begin{align*}
 f(x) &= \frac{\ln x}{x} \\
 f'(x) &= \frac{d}{dx} \left[ \frac{\ln x}{x} \right] \\
 &= \frac{(x) \frac{d}{dx} [\ln x] - \ln x \frac{d}{dx} [x]}{(x)^2} \\
 &= \frac{(x) \left( \frac{1}{x} \right) - \ln x (1)}{x^2} \\
 &= \frac{1 - \ln x}{x^2}
\end{align*}
\]

For \(f'(c) = 0\), the numerator must equal zero,

\[
\begin{align*}
 f'(c) = 0 &= \frac{1 - \ln c}{c^2} \\
 0 &= 1 - \ln c \\
 \ln c &= 1 \\
 c &= e
\end{align*}
\]

We have shown that \(f'(e) = 0\). The critical number is \(e\), which lies in the interval \((2, 3)\).
Now, we evaluate the function at the critical numbers in the interval and at the endpoints of the interval:

\[
\begin{align*}
    f(e) &= \frac{\ln e}{e} = \frac{1}{e} \\
    f(1) &= \frac{\ln 1}{1} = 0 \\
    f(3) &= \frac{\ln 3}{3}
\end{align*}
\]

The smallest number is 0, so this is the absolute min and it occurs at \(x = 1\).

It is difficult to determine if \(1/e > \ln 3/3\) without resorting to a calculator, or more powerful techniques we have yet to learn. But we can argue based on the properties of derivatives that we must have \(1/e > \ln 3/3\).

Since the function is continuous in the interval and has a minimum at \(x = 1\), and the derivative at \(x = e\) is zero means the tangent line is horizontal at \(x = e\), and there are no other critical numbers for the function, the function must lie below its tangent line near \(x = e\). Read that again and see why are the conditions listed are necessary. You might want to draw the function above the tangent line at \(x = e\) and see how that leads to contradictions. The function must therefore look something like:

![Graph showing the function and tangent line at x = e](image)

Therefore, the function has an absolute max of \(1/e\) at \(x = e\).

**Example** If \(a\) and \(b\) are positive numbers, find the maximum value of \(f(x) = x^a(1-x)^b\), \(0 \leq x \leq 1\).

The derivative will always exist since \(a\) and \(b\) are positive (if they could be negative, we could have a denominator other than 1).
The only critical numbers will be if \( f'(c) = 0 \):

\[
f(x) = x^a(1 - x)^b \\
f'(x) = \frac{d}{dx} [x^a(1 - x)^b] \\
= x^a \frac{d}{dx} [(1 - x)^b] + (1 - x)^b \frac{d}{dx} [x^a] \\
= bx^a(1 - x)^{b-1}(-1) + a(1 - x)^b x^{a-1} \\
= a(1 - x)^b x^{a-1} - bx^a(1 - x)^{b-1} \\
= (1 - x)^b x^{a-1} (a - b(1 - x)^{-1}) \\
= (1 - x)^b x^{a-1} \left( \frac{a}{x} - \frac{b}{1-x} \right) \\
= (1 - x)^b x^{a-1} \left( \frac{a(1 - x) - bx}{x(1 - x)} \right) \\
= (1 - x)^b x^{a-1} (a - (a + b)x) 
\]

The critical number is therefore \( c = \frac{a}{a + b} \).

Now, we evaluate the function at the critical numbers in the interval and at the endpoints of the interval:

\[
f \left( \frac{a}{a + b} \right) = \left( \frac{a}{a + b} \right)^a \left( 1 - \left( \frac{a}{a + b} \right) \right)^b \\
= \left( \frac{a}{a + b} \right)^a \left( \frac{b}{a + b} \right)^b > 0 \text{ (since } a \text{ and } b \text{ are positive)} \\
f(0) = 0^a(1 - 0)^b = 0 \\
f(1) = 1^a(1 - 1)^b = 0 
\]

Therefore, the maximum value of \( f(x) = x^a(1 - x)^b \), \( 0 \leq x \leq 1 \) is \( \left( \frac{a}{a + b} \right)^a \left( \frac{b}{a + b} \right)^b \) which occurs at \( x = \frac{a}{a + b} \).