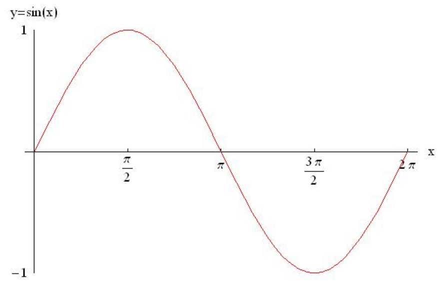
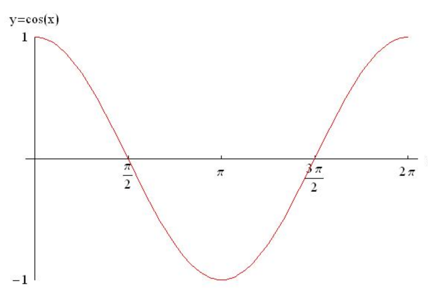
1011 Precalculus Lecture 4.5

The Sine Function



Domain: $x \in \mathbb{R}$
Range: $y \in [-1, 1]$
Continuity: continuous for all x
Increasing-decreasing behaviour: alternately increasing and decreasing
Symmetry: odd $(\sin(-x) = -\sin(x)))$
Boundedness: bounded above and below
Local Extrema: absolute max of $y = 1$, absolute min of $y = -1$
Horizontal Asymptotes: none
Vertical Asymptotes: none
End behaviour: The limits as x approaches $\pm\infty$ do not exist since the function values oscillate between $+1$ and -1.
This is a periodic function with period 2π.

The Cosine Function



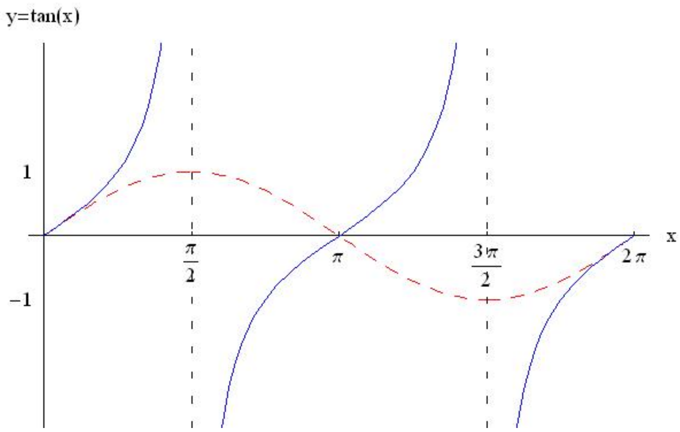
Domain: $x \in \mathbb{R}$
Range: $y \in [-1, 1]$
Continuity: continuous for all x
Increasing-decreasing behaviour: alternately increasing and decreasing
Symmetry: even $(\cos(-x) = \cos(x)))$
Boundedness: bounded above and below
Local Extrema: absolute max of $y = 1$, absolute min of $y = -1$
Horizontal Asymptotes: none
Vertical Asymptotes: none
End behaviour: The limits as x approaches $\pm\infty$ do not exist since the function values oscillate between $+1$ and -1.
This is a periodic function with period 2π.

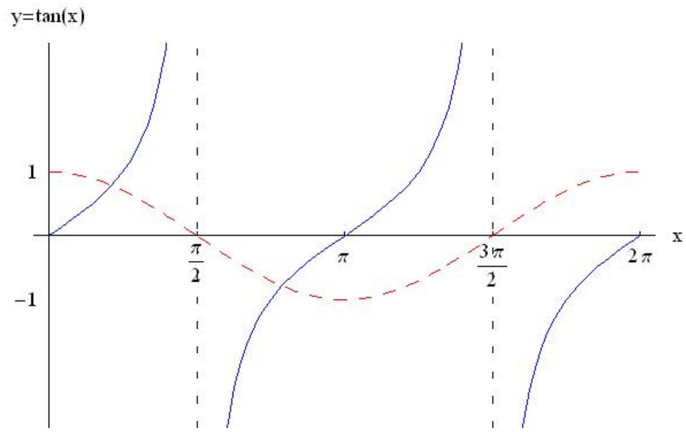
The Tangent Function

The tangent function is $\tan x = \dfrac{\sin x}{\cos x}$.

It will have zeros where the sine function has zeros, and vertical asymptotes where the cosine function has zeros.

It will look like the sine function where the cosine is essentially equal to 1, which is when x is near 0 or 2π.





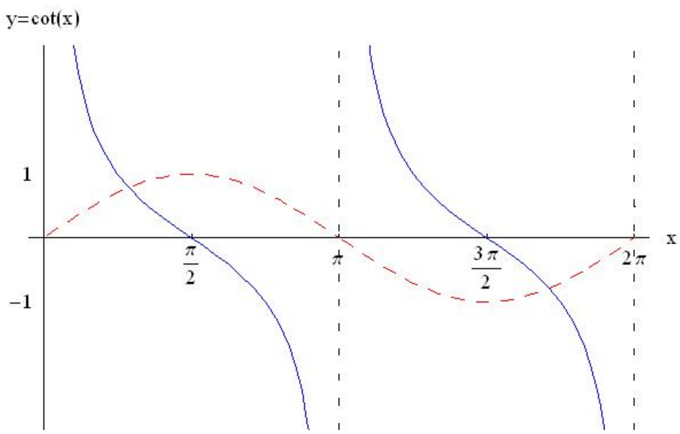
Domain: $x \in \mathbb{R}$ except $x = \dfrac{\pi}{2} + k\pi$, $k = \ldots, -3, 2, 1, 0, 1, 2, 3, \ldots$
Range: $y \in \mathbb{R}$
Continuity: continuous on its domain
Increasing-decreasing behaviour: increasing on each interval in its domain
Symmetry: odd $(\tan(-x) = -\tan(x)))$
Boundedness: not bounded
Local Extrema: none
Horizontal Asymptotes: none
Vertical Asymptotes: $x = \dfrac{\pi}{2} + k\pi$, $k = \ldots, -3, 2, 1, 0, 1, 2, 3, \ldots$
End behaviour: The limits as x approaches $\pm\infty$ do not exist since the function values oscillate between $-\infty$ and $+\infty$.
This is a periodic function with period π.

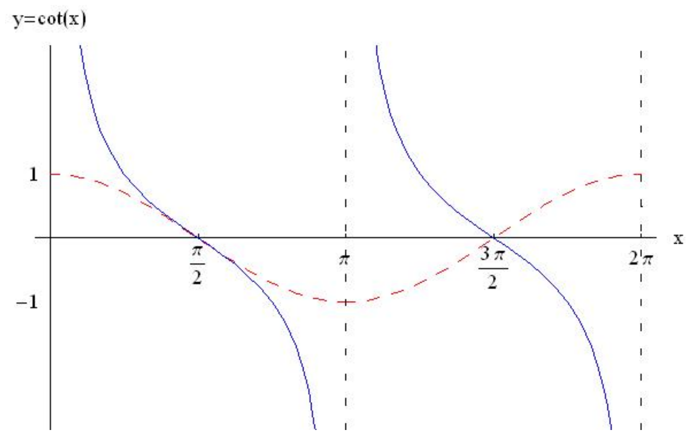
The Cotangent Function

The tangent function is $\cot x = \dfrac{\cos x}{\sin x}$.

It will have zeros where the cosine function has zeros, and vertical asymptotes where the sine function has zeros.

It will look like the cosine function where the sine is essentially equal to 1, which is when x is near $\pi/2$.



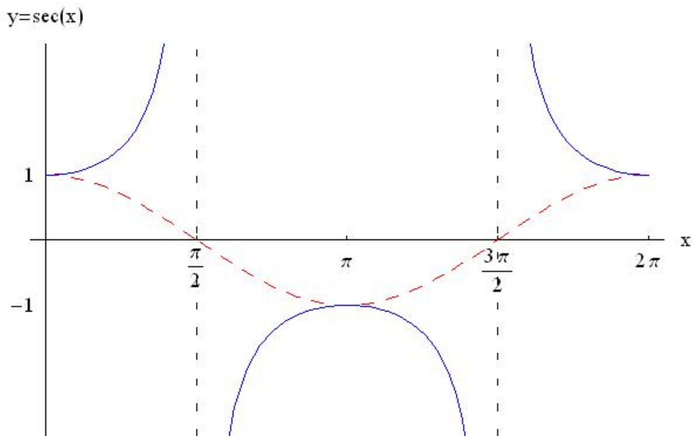


Domain: $x \in \mathbb{R}$ except $x = k\pi$, $k = \ldots, -3, 2, 1, 0, 1, 2, 3, \ldots$
Range: $y \in \mathbb{R}$
Continuity: continuous on its domain
Increasing-decreasing behaviour: decreasing on each interval in its domain
Symmetry: odd $(\cot(-x) = -\cot(x)))$
Boundedness: not bounded
Local Extrema: none
Horizontal Asymptotes: none
Vertical Asymptotes: $x = k\pi$, $k = \ldots, -3, 2, 1, 0, 1, 2, 3, \ldots$
End behaviour: The limits as x approaches $\pm\infty$ do not exist since the function values oscillate between $-\infty$ and $+\infty$.
This is a periodic function with period π.

The Secant Function

The secant function is $\sec x = \dfrac{1}{\cos x}$.

It will have vertical asymptotes where the cosine function has zeros. It will have no zeros.

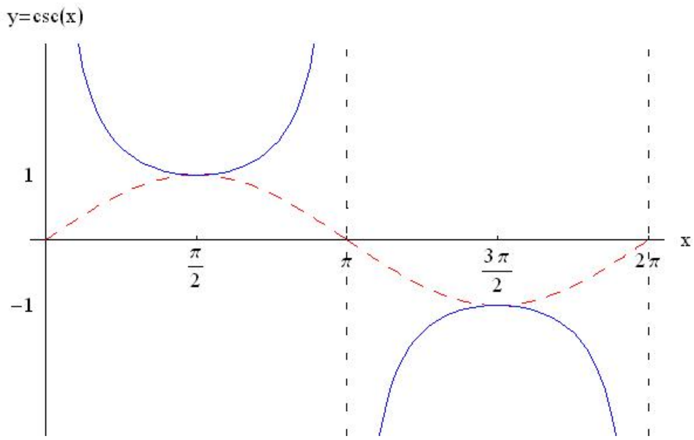


Domain: $x \in \mathbb{R}$ except $x = \dfrac{\pi}{2} + k\pi,\ k = \ldots, -3, 2, 1, 0, 1, 2, 3, \ldots$
Range: $y \in (-\infty, -1] \cup [1, \infty)$
Continuity: continuous on its domain
Increasing-decreasing behaviour: increases and decreases on each interval in its domain
Symmetry: even ($\sec(-x) = \sec(x)$))
Boundedness: not bounded
Local Extrema: local min at $x = 2k\pi$, local max at $x = (2k+1)\pi,\ k = \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$
Horizontal Asymptotes: none
Vertical Asymptotes: $x = \dfrac{\pi}{2} + k\pi,\ k = \ldots, -3, 2, 1, 0, 1, 2, 3, \ldots$
End behaviour: The limits as x approaches $\pm\infty$ do not exist since the function values oscillate between $-\infty$ and $+\infty$.
This is a periodic function with period 2π.

The Cosecant Function

The cosecant function is $\csc x = \dfrac{1}{\sin x}$.

It will have vertical asymptotes where the sine function has zeros. It will have no zeros.

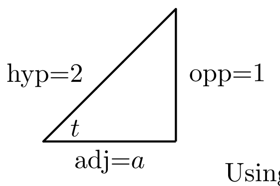


Domain: $x \in \mathbb{R}$ except $x = k\pi$, $k = \ldots, -3, 2, 1, 0, 1, 2, 3, \ldots$
Range: $y \in (-\infty, -1] \cup [1, \infty)$
Continuity: continuous on its domain
Increasing-decreasing behaviour: increases and decreases on each interval in its domain
Symmetry: odd $(\csc(-x) = -\csc(x)))$
Boundedness: not bounded
Local Extrema: local min at $x = \pi/2 + 2k\pi$, local max at $x = 3\pi/2 + 2k\pi$, $k = \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$
Horizontal Asymptotes: none
Vertical Asymptotes: $x = k\pi$, $k = \ldots, -3, 2, 1, 0, 1, 2, 3, \ldots$
End behaviour: The limits as x approaches $\pm\infty$ do not exist since the function values oscillate between $-\infty$ and $+\infty$.
This is a periodic function with period 2π.

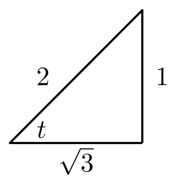
Example 4.5.30 Solve the equation $\csc t = 2$ in the interval $\pi/2 \le t \le \pi$. You should not need a calculator to solve this problem.

Solve this using reference triangles. The cosecant is positive where the sine is positive, which means we are in either Quadrant I or II. With no other information, this is all we can say. Let's assume we are in Quadrant I and proceed with our solution. Since we are told not to use a calculator, we expect that the reference triangle will be one of the two special triangles, a 45-45-90 or a 30-60-90, or the angle will be a quadrantal angle.

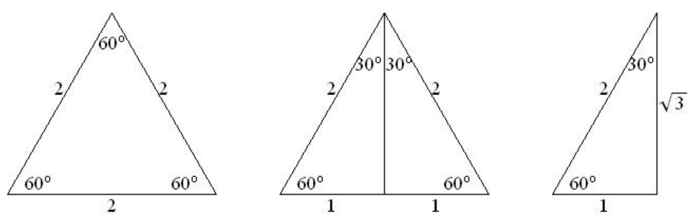
$$\csc t = 2 = \frac{2}{1} = \frac{\text{hyp}}{\text{opp}}$$



Using the Pythagorean Theorem, we see $2^2 = a^2 + 1^2 \longrightarrow a = \sqrt{3}$.

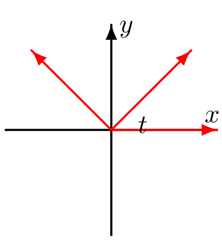


This triangle results from the equilateral triangle:



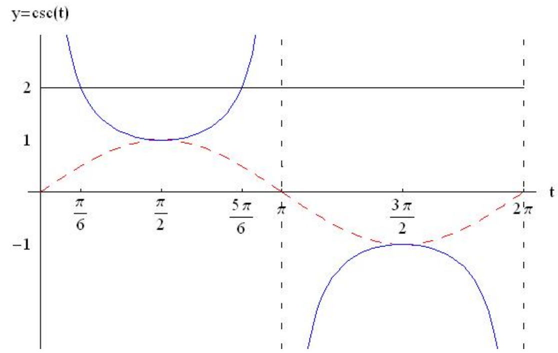
We see the angle is $30^\text{o} = \pi/6$ radians. The cosecant is periodic with period 2π, so another solution is $t = \pi/6 + 2\pi = 13\pi/6$. This is also not in the interval asked for.

We still have a solution in Quadrant II, which we haven't dealt with yet. Let's do that now.



There is a solution in Quadrant II: $\pi - \pi/6 = 5\pi/6$. Since this is in the interval asked for, we have found the solution.

Here is a sketch of the cosecant function so you can see the solution makes sense

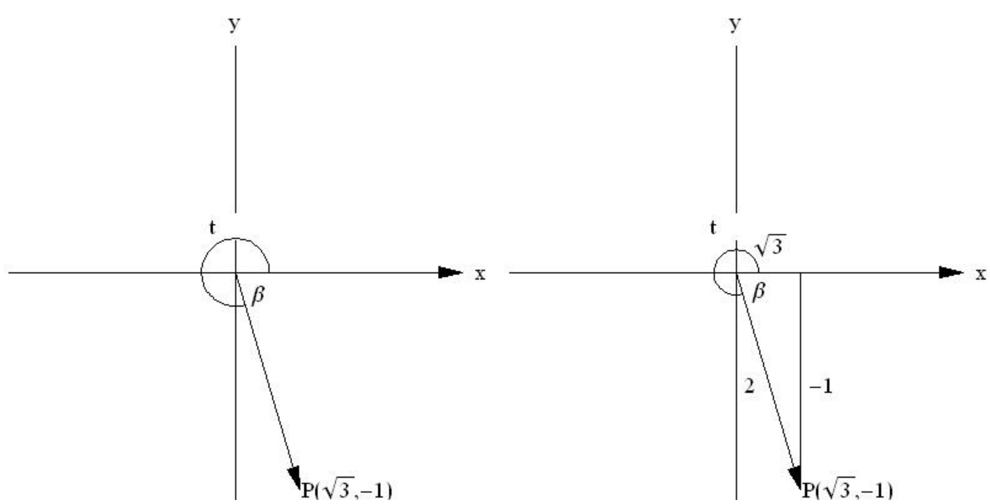


Example Solve the equation $\cot t = -\sqrt{3}$ for all possible values of t.

Let's use the "point in the plane" (sometimes I call the circle method) to solve the problem. When the cotangent of t is less than zero we must be in either Quadrant II or Quadrant IV (where $x = r\cos t$ and $y = r\sin t$ have different signs). With no other information provided, this is all we can say.

Let's assume we are in the fourth quadrant, and see what happens. Quadrant IV means $y < 0$.

$$\cot t = -\sqrt{3} = \frac{\sqrt{3}}{-1} = \frac{x}{y}$$



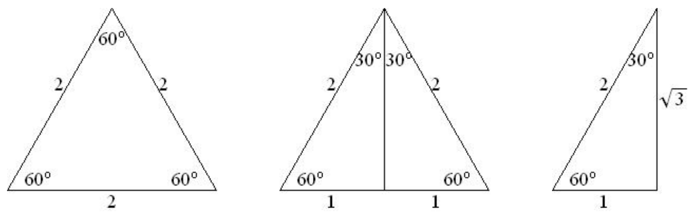
In the second diagram, I have labelled the lengths of the sides, being careful to indicate quantities which are less than zero based on the quadrant the point P is in.

Also, we have found r using the Pythagorean theorem.

$$r = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2.$$

We see from the diagram that this is one of our special triangles, a 30-60-90 triangle.

This triangle results from the equilateral triangle:



We see the angle $\beta = 30^{\circ} = \pi/6$ radians.

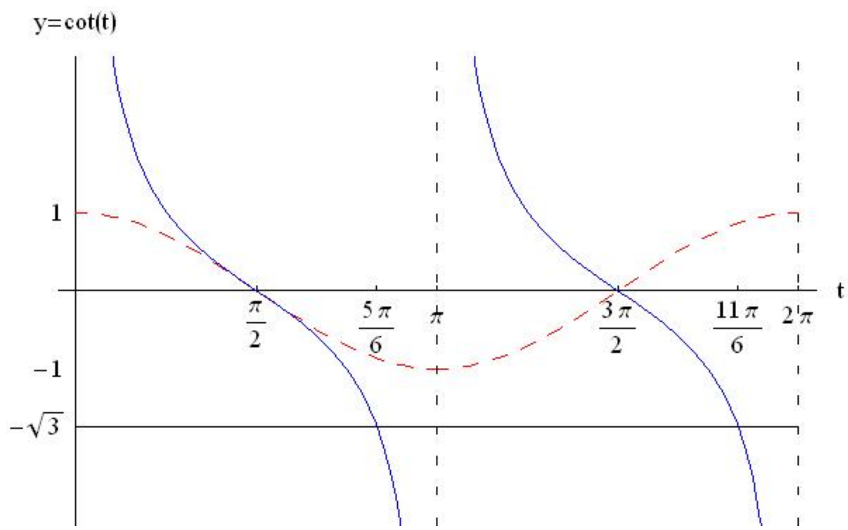
The angle we seek is $t = 2\pi - \beta = 2\pi - \pi/6 = 11\pi/6$.

Since the cotangent function has period π, all the solutions t to the equation $\cot t = -\sqrt{3}$ are

$$t = \frac{11\pi}{6} + k\pi, \quad k = 0, \pm 1, \pm 2, \pm 3, \ldots.$$

Notice that $k = \pm 1$ gives us the solution ($t = 5\pi/6$) we would have found had we chosen to work in Quadrant II.

Here is a sketch of the cotangent function so you can see the solution makes sense



Notice that in the solution, I used both a reference triangle as well as the x, y, r formalism. Use whatever seems best at the time, and ensure that your solution is mathematically consistent.