Example 1.5.6 Given the parametric equations

$$x = t + 1, \quad y = t^2 - 2t.$$

Find the points determined by $t = -3, -2, -1, 0, 1, 2, 3$.
Find a direct relationship between x and y. Is this an explicit function relationship?
Graph the relationship in the xy-plane.

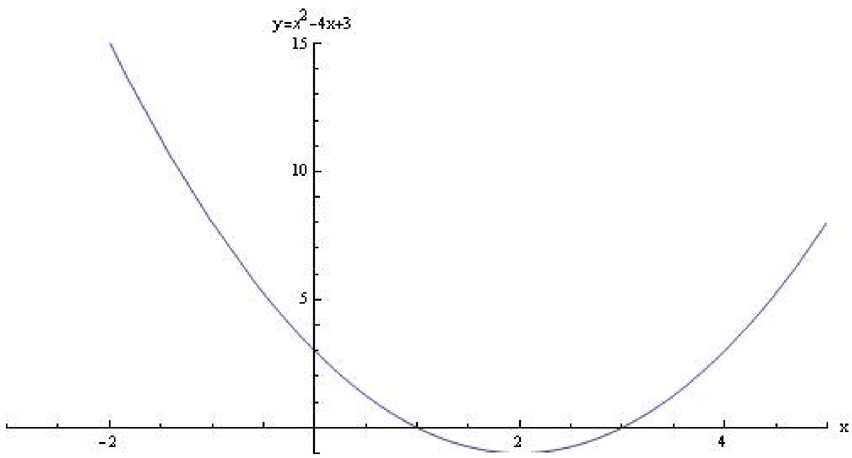
t	$x = t + 1$	$y = t^2 - 2t$
-3	-2	15
-2	-1	8
-1	0	3
0	1	0
1	2	-1
2	3	0
3	4	3

To find a direct relationship, we need to eliminate the parameter t from the two equations.

Solve $x = t + 1$ for t: $t = x - 1$.
Substitute $t = x - 1$ into the equation $y = t^2 - 2t$:

$$\begin{aligned} y &= t^2 - 2t \\ &= (x-1)^2 - 2(x-1) \\ &= x^2 - 2x + 1 - 2x + 2 \\ &= x^2 - 4x + 3 \end{aligned}$$

The function is a quadratic. Here is a sketch I created using *Mathematica*:



Notice the points on the sketch agree with the table we created above.

Example 1.5.17 Find a formula for $f^{-1}(x)$ given $f(x) = \sqrt{x-3}$. Give the domain of $f^{-1}(x)$.

$$\begin{aligned}
y &= \sqrt{x-3} \quad \text{Step 1: let } y = f(x) \\
x &= \sqrt{y-3} \quad \text{Step 2: Flip } x \text{ and } y \\
x^2 &= (y-3) \quad \text{Step 3: Solve for } y \\
x^2 + 3 &= y \\
y &= x^2 + 3 \\
f^{-1}(x) &= x^2 + 3 \quad \text{Finally, } f^{-1}(x) = y
\end{aligned}$$

The function f has a range of $y \in [0, \infty)$. This imposes a restriction on the domain of $f^{-1}(x)$. Since $x^2 + 3$ has domain $x \in (-\infty, \infty)$, the domain of $f^{-1}(x)$ is $x \in [0, \infty)$.

Example 1.5.22 Find a formula for $f^{-1}(x)$ given $f(x) = (x-2)^{1/3}$. Give the domain of $f^{-1}(x)$.

$$\begin{aligned}
y &= (x-2)^{1/3} \quad \text{Step 1: let } y = f(x) \\
x &= (y-2)^{1/3} \quad \text{Step 2: Flip } x \text{ and } y \\
x^3 &= (y-2) \quad \text{Step 3: Solve for } y \\
x^3 + 2 &= y \\
y &= x^3 + 2 \\
f^{-1}(x) &= x^3 + 2 \quad \text{Finally, } f^{-1}(x) = y
\end{aligned}$$

Since there is no restriction imposed by f, the domain of $f^{-1}(x)$ is $x \in (-\infty, \infty)$.